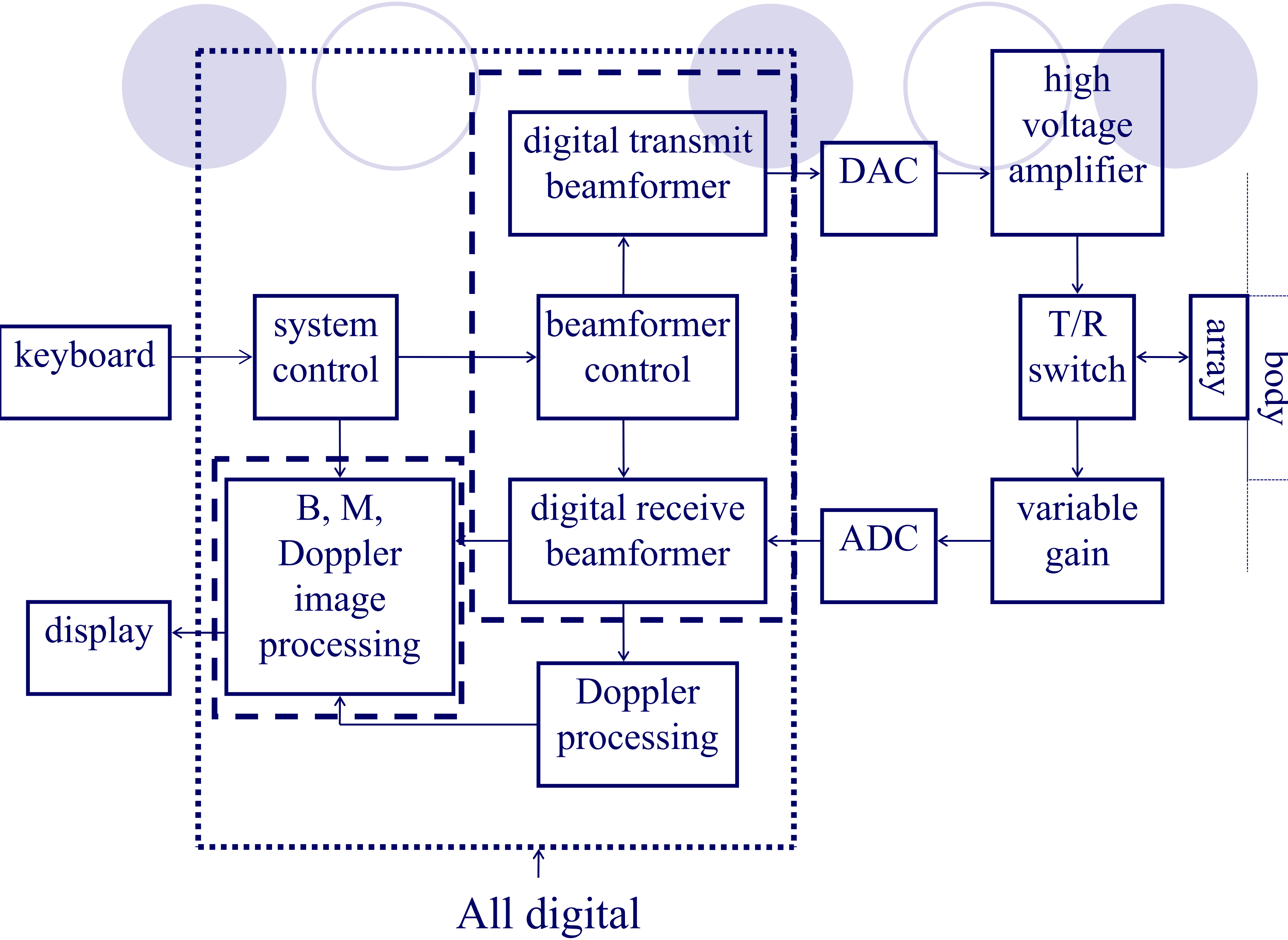
Digital Signal Processing-

Basic Concepts

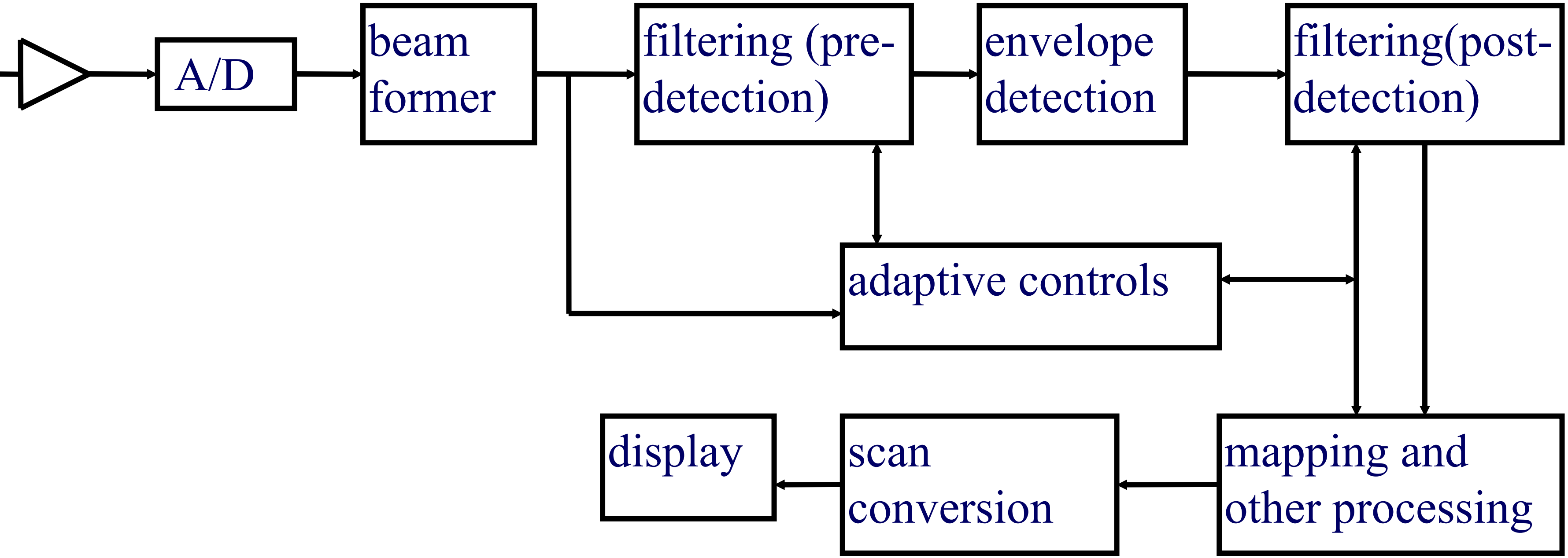
Some materials from Dr. Larry Marple are acknowledged

Outline

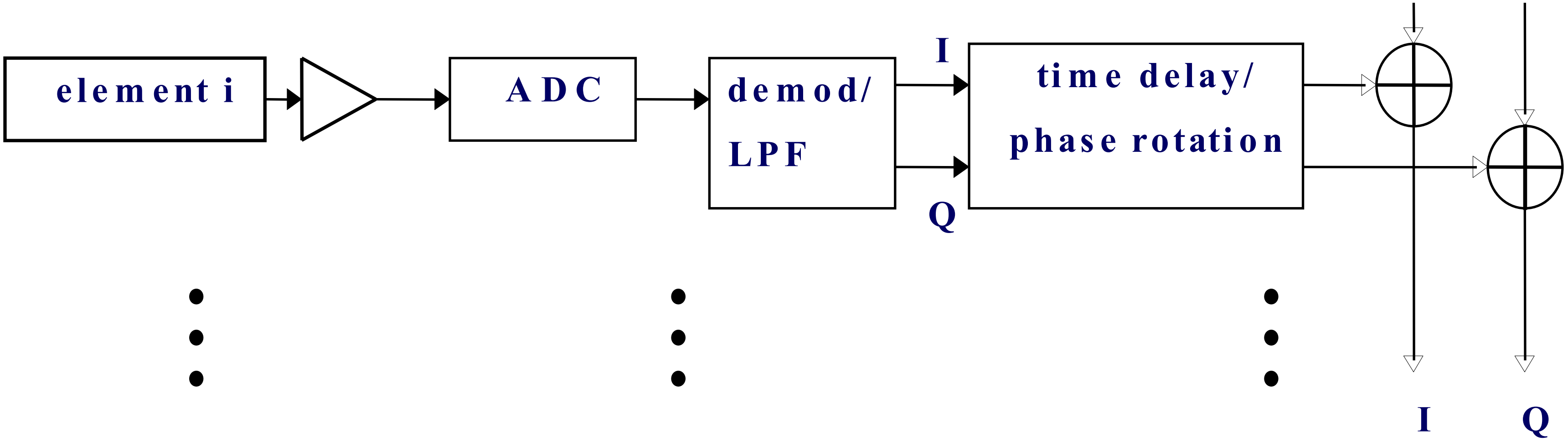
- Background materials
- Continuous-time signals and transforms
- Fourier transform and properties
- Sampling and windowing operations
- Discrete-time signals and transforms


digital transmit beamformer
DAC
high voltage amplifier
keyboard
system control
beamformer control
T/R switch
array
body
B, M, Doppler image processing
digital receive beamformer
ADC
variable gain
display
Doppler processing
All digital

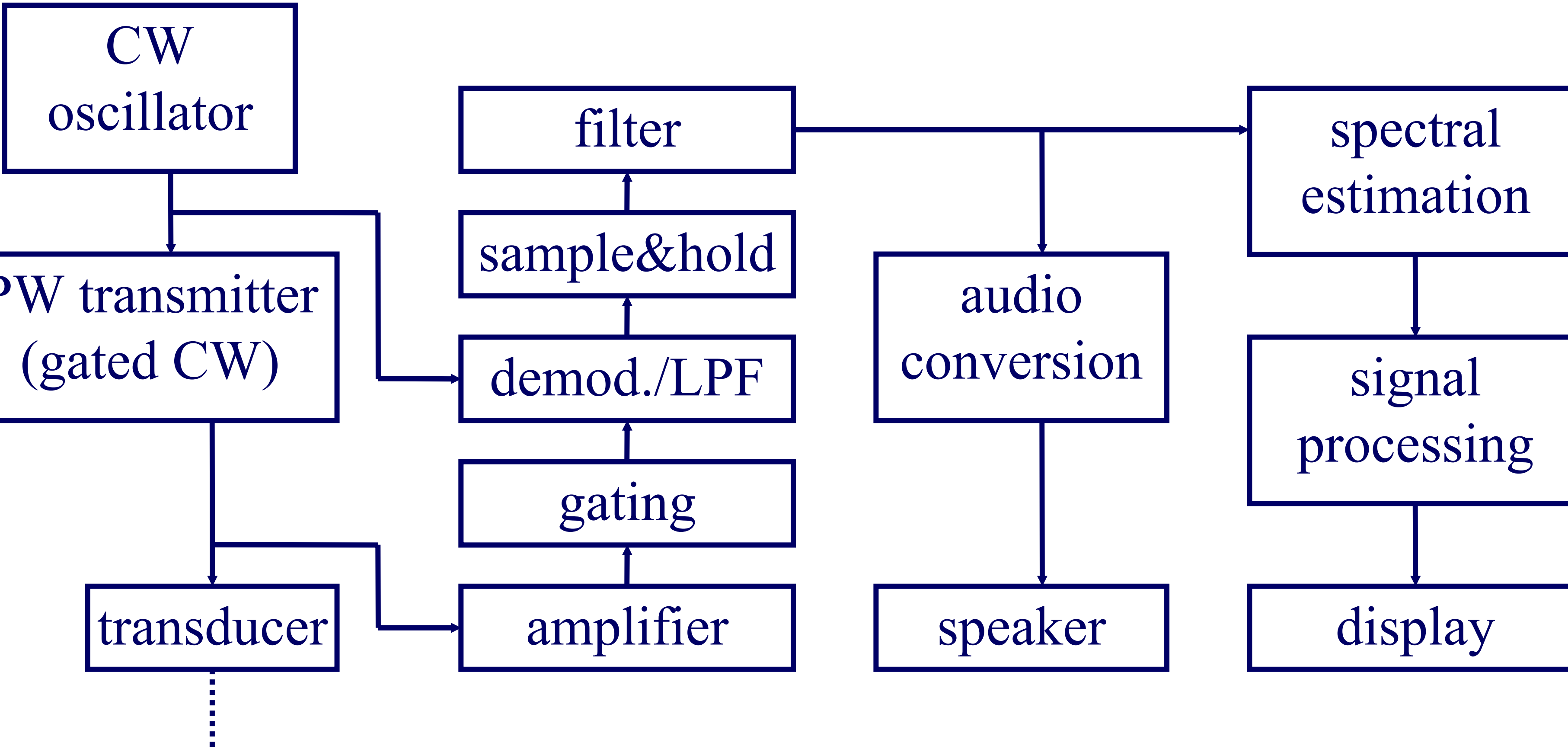
Generic Receiver



Baseband Beamformer



PW Doppler

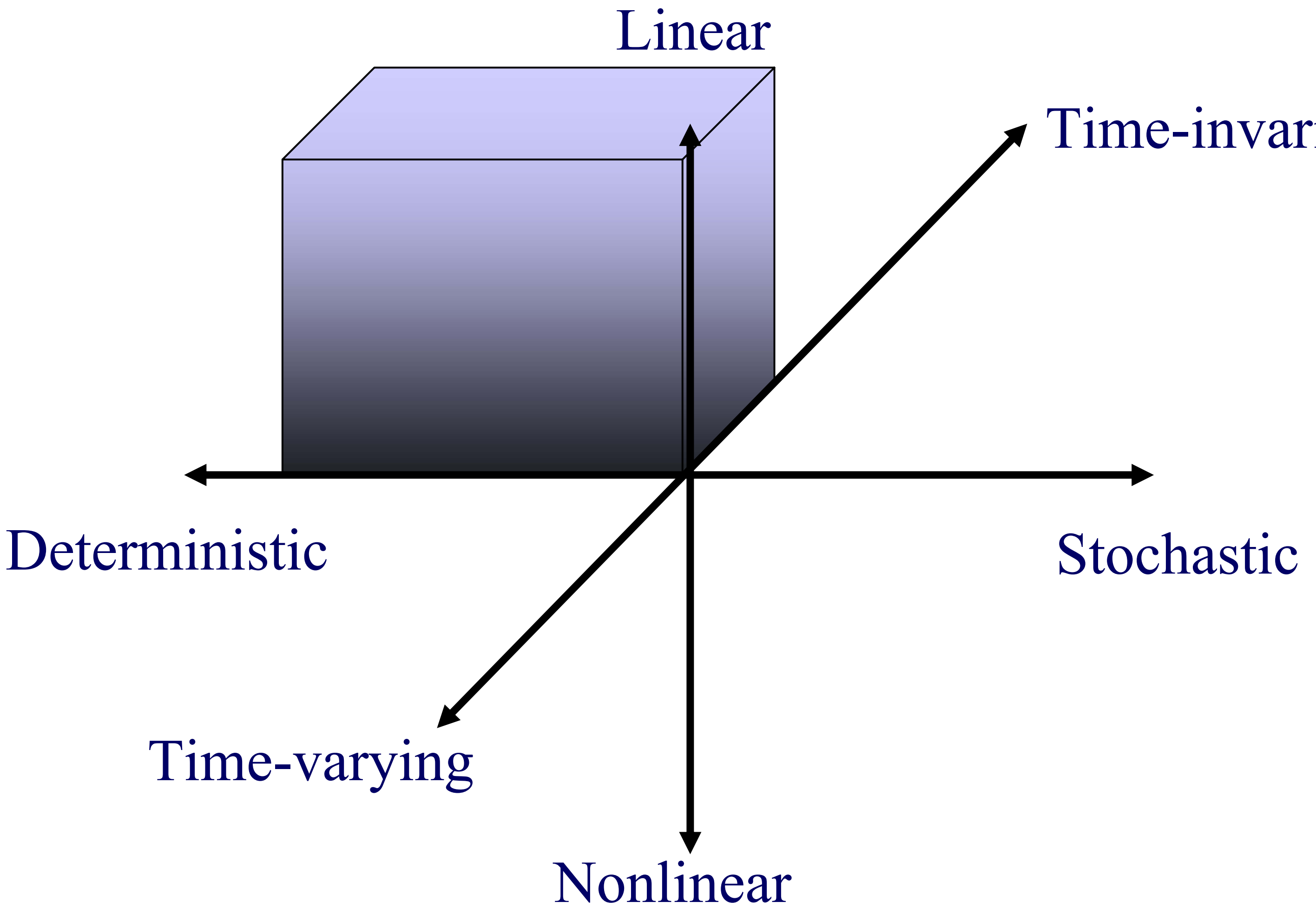


CW oscillator
filter
spectral estimation
sample&hold
PW transmitter (gated CW)
audio conversion
demod./LPF
signal processing
gating
transducer
amplifier
speaker
display

DSP Building Blocks in Imaging

- Filter
- Modulator/demodulator
- Decimator/interpolator
- Fourier transformer
- Hilbert transformer
- Detector
- Autocorrelator
- Beamformer

Hierarchy of Signal & System Properties



Temporal & Spatial Signal Distinction

- Continuous-time (CT) signals
  Continuous-space (CS) signals
  → Analog signals
- Discrete-time (DT) signals
  Discrete-space (DS) signals
  → Sampled-data signals (typically uniform sampling)
- Digital signals: DT or DS signals with quantized ampltitudes

Main Approach

- All DT (DS) DSP theory is derived from the CT (CS) signal processing theory.
- Sampling scheme: one value per sampling point.
- Sampling scheme: uniform temporal (spatial) sampling intervals (will introduce periodicity).

$$x[n] = x(nT) \qquad T: \text{Sampling interval (sec)}$$
$$y[m] = x(mD) \qquad D: \text{Sampling interval (m)}$$

Signal Representation Domains

- *t* (time, sec) ←→ *f* (temporal frequency, cycles/sec=Hertz)
- *d* (space, m) ←→ *k* (spatial frequency, wavenumber, cycles/meter)

  or $\lambda=1/k$ (wavelength, meters/cycle)
- (*t*, *d*) (propagating waves, coupled time and space signal) ←→ (*f*, *k*) Coupled frequencies ($k=f/c$)

Acoustic Wave Equations

$$\frac{\partial^2 w(z,t)}{\partial t^2} = (B/\rho)\frac{\partial^2 w(z,t)}{\partial z^2}$$

$$w(z,\omega) = w_1(\omega)e^{-j\omega z/c} + w_2(\omega)e^{j\omega z/c}$$

$$w(z,t) = w_1(t - z/c) + w_2(t + z/c)$$

Complex Numbers and Complex Arithmetic

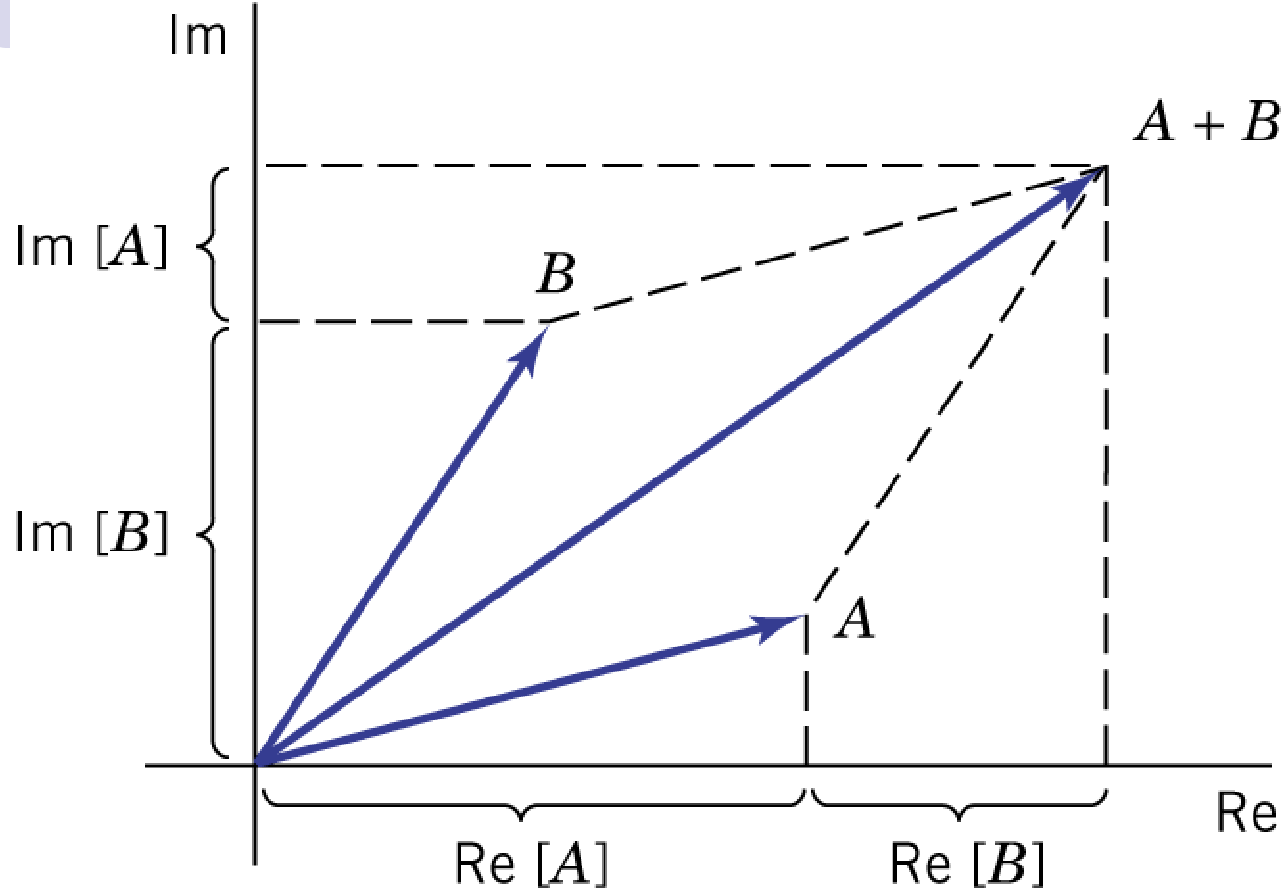
- Required to define roots of polynomials:

$$z^2 + 1 = 0$$

- Required to define solutions of linear differential (difference) equations:

$$\frac{d^2 X(s)}{ds^2} + X(s) = 0$$

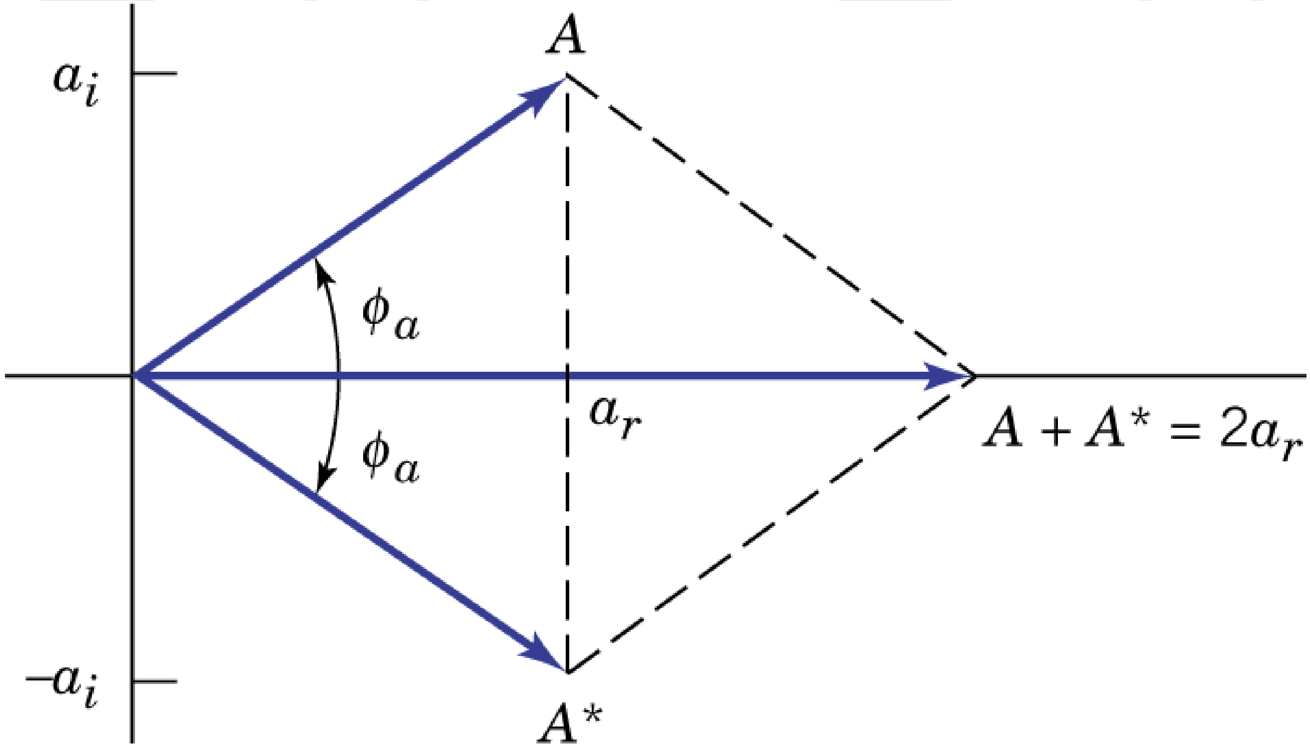
Complex Addition and Subtraction



$$\mathrm{Re}[A \pm B] = \mathrm{Re}[A] \pm \mathrm{Re}[B]$$

$$\mathrm{Im}[A \pm B] = \mathrm{Im}[A] \pm \mathrm{Im}[B]$$

Complex Conjugate



$$A = |A|\angle\phi_a = a_r + j \cdot a_i$$

$$A^* \equiv |A|\angle -\phi_a = a_r - j \cdot a_i$$

$$A \cdot A^* = a_r^2 + a_i^2 = |A|^2$$

Complex Multiplication

$$A \cdot B = (a_r b_r - a_i b_i) + j \cdot (a_r b_i - a_i b_r) = \mathrm{Re}\left[AB\right] + \mathrm{Im}\left[AB\right]$$

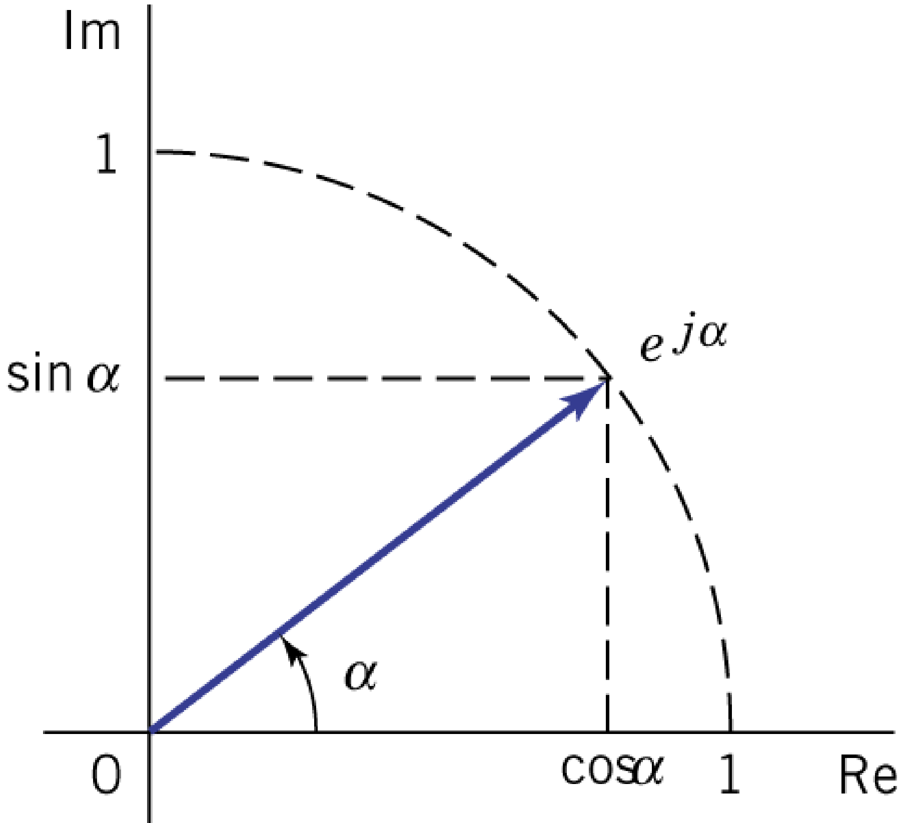
$$\text{If } k \text{ is real, } \mathrm{Re}[kB] = k\,\mathrm{Re}[B], \mathrm{Im}[kB] = k\,\mathrm{Im}[B]$$

Complex Division (Rationalization)

$$\frac{B}{A} = \frac{BA^*}{AA^*} = \frac{b_r a_r + b_i a_i}{a_r^2 + a_i^2} + j\frac{b_r a_i - b_i a_r}{a_r^2 + a_i^2}$$

Complex Number in Exponential Form

- Euler's formula: $e^{\pm j\alpha} = \cos\alpha \pm j\sin\alpha$
- Complex number in exponential form: $A = |A|e^{j\phi_a}$



$$A \cdot B = |A||B| \angle \phi_a + \phi_b$$

$$\frac{A}{B} = \frac{|A|}{|B|} \angle \phi_a - \phi_b$$

$$j = 1\angle 90^0$$

Complex Signals

- Signals represented as a pair of linked real-valued signals:

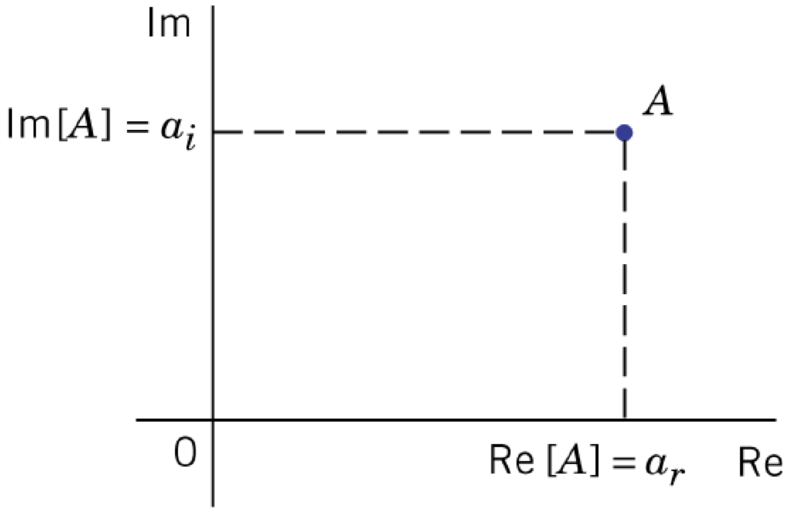
$$y(t) = \{y_r(t), y_i(t)\} = y_r(t) + j \cdot y_i(t)$$

$$y_r(t) : \text{real part or in - phase component (I)}$$

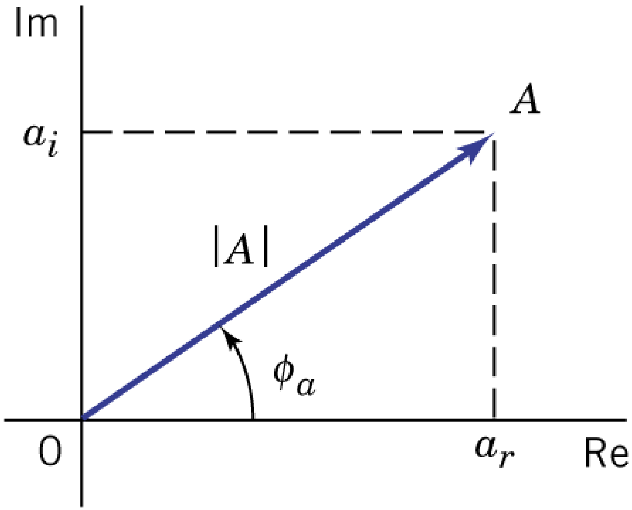
$$y_i(t) : \text{imaginary part or quadrature - phase component (Q)}$$

Complex Numbers in the Complex Plane

$j \equiv \sqrt{-1}$



(a) The complex plane with
$A = a_r + ja_i$



(b) Polar coordinates for
$A = |A| \angle \phi_a$

$$a_r = |A| \cos \phi_a$$

$$a_i = |A| \sin \phi_a$$

$$|A| = \left(a_r^2 + a_i^2\right)^{1/2}$$

$$\phi_a = \tan^{-1}\left(\frac{a_i}{a_r}\right) \quad a_r > 0$$

$$\phi_a = \pm 180^0 - \tan^{-1}\left(\frac{a_i}{a_r}\right) \quad a_r < 0$$

I/Q signals are 90° out of phase

$$\sin(\theta(t) + 90^0) = \cos(\theta(t))$$

Complex Signals

- Created from real-valued signals by operations such as
  - Complex modulation/demodulation (aka quadrature modulation)
  - Fourier transformation
  - Complex filtering
  - Baseband signal generation from memory
- Advantages
  - Simplifies mathematical analysis
  - Reduce hardware data rates
  - Reduces arithmetic and filtering requirements for modulation/demodulation/phase adjustment

Advantage of Complex Representation: Modulation as an Example

$$x_1(t) = a_1(t)\cdot e^{\theta_1(t)}, \quad x_2(t) = a_2(t)\cdot e^{\theta_2(t)}$$

$$x_1(t)\cdot x_2(t) = \left[a_1(t)\cdot a_2(t)\right]e^{\theta_1(t)+\theta_2(t)}$$

or

$$x_1(t) = a_1(t)\cdot \cos\theta_1(t), \quad x_2(t) = a_2(t)\cdot \cos\theta_2(t)$$

$$x_1(t)\cdot x_2(t) = \frac{1}{2}\left[a_1(t)\cdot a_2(t)\right]\left[\cos\left(\theta_1(t)-\theta_2(t)\right)+\cos\left(\theta_1(t)+\theta_2(t)\right)\right]$$

Arbitrary Real Signal

- Exponential argument is an arbitrary function of t

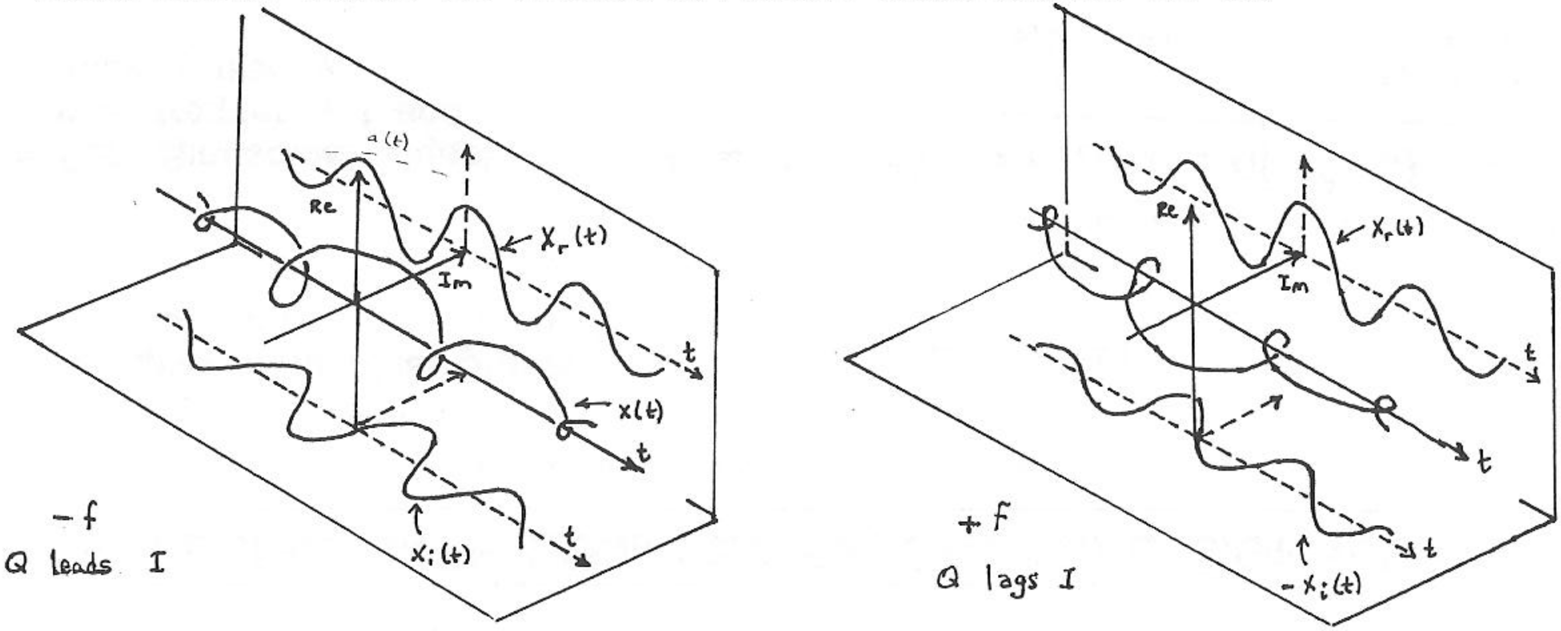
$$x(t) = \frac{1}{2}a(t)e^{j\theta(t)} + \frac{1}{2}a(t)e^{-j\theta(t)}$$

↓ analytic signal ↓ conjugate analytic signal

$a(t)$: envelop function

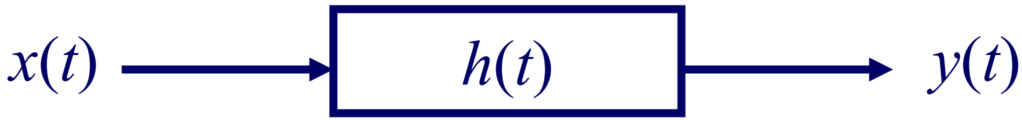
Do Negative Frequencies Exist?

COMPONENT LAGS OR LEADS IN-PHASE COMPONENT BY 90°



Continuous-Time Signals and Transforms

Continuous-Time System Response



$$y(t)=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau=\int_{-\infty}^{\infty}h(t-\tau)x(\tau)d\tau$$

$$=h(t)*x(t)=x(t)*h(t)$$

- Consider responses to the following inputs
  - Complex exponential signal
  - Complex sinusoidal signal
  - Modulated Gaussian signal
  - Impulse function (limit of Gaussian signals)

Continuous-Time System Response to Complex Exponential

- Motivation for Laplace transform

$$\text{Let } x(t) = e^{st}, s = \sigma + j\omega$$

$$y(t) = \int_{-\infty}^{t} h(\tau) e^{s(t-\tau)} d\tau = e^{st} H(s),$$

where $H(s)$ is the Laplace transform of $h(t)$

for limits and s where integral exists

- e^{st} is an eigenfunction of the LTI CT system
- $H(s)$ is the continuous-time system function

Continuous-Time System Response to Complex Sinusoidal Signal

- Motivation for Fourier Transform

$$\text{Let } s = 0 + j\omega, x(t) = e^{st} = e^{j\omega t} = \cos\omega t + j\sin\omega t$$

$$H(s = j\omega) = H(\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$

where $H(\omega)$ is the Fourier transform of $h(t)$

- Fourier transform of the system temporal function is the system response function to input complex sinusoidal signals

$$LT = FT\left\{h(t)e^{\sigma t}\right\}$$

Gaussian Signals

- Unmodulated Gaussian Signal (the transform is also Gaussian)

$$g(t) = e^{-\pi t^2/T^2} \Leftrightarrow G(f) = Te^{-\pi f^2 T^2}$$

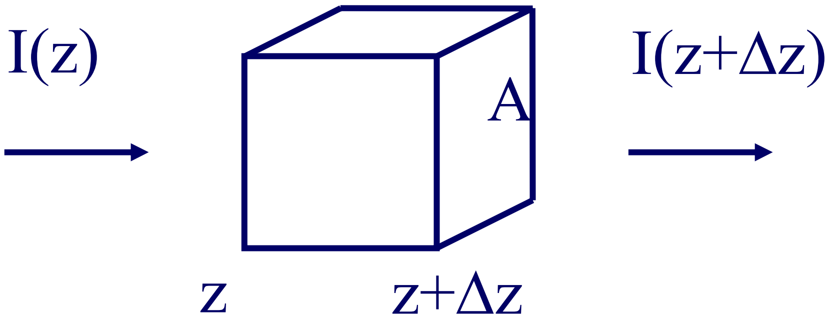
- Modulated Gaussian Signal (with complex sinusoid)

$$x(t) = g(t)Ae^{j2\pi f_c t} \Leftrightarrow X(f) = G(f) * A\delta(f - f_c)$$

$$= ATe^{-\pi(f-f_c)^2 T^2}$$

Body Ultrasound Attenuation Filter

- Frequency domain attenuation response



$$A \cdot I(z+\Delta z) = A \cdot I(z) - 2\beta A \cdot I(z)\Delta z$$

$$-\frac{\partial I(z)}{\partial z} = 2 \cdot \beta I(z)$$

$$I(z) = I_0 e^{-2\beta z}$$

$$\beta = \alpha f$$

Body Ultrasound Attenuation Filter

$$H(z,f) = e^{-(\alpha f z + j2\pi f z / c)}$$

$$I(z,f) = I_0 \left| H(z,f) \right|^2 = I_0 e^{-2\alpha f z}$$

Body Ultrasound Attenuation Filter

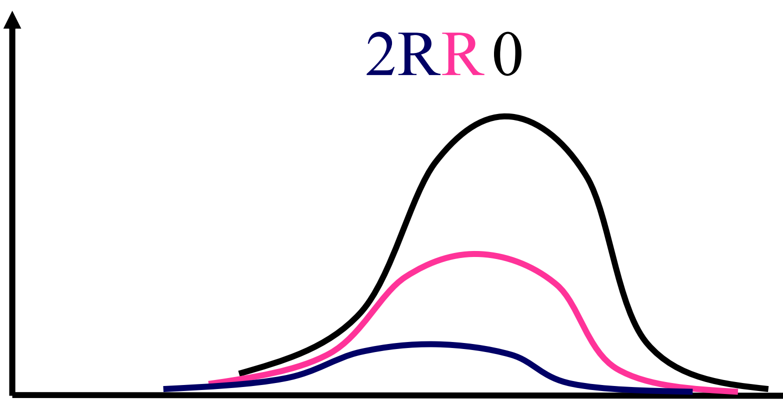
- Assuming a Gaussian signal:

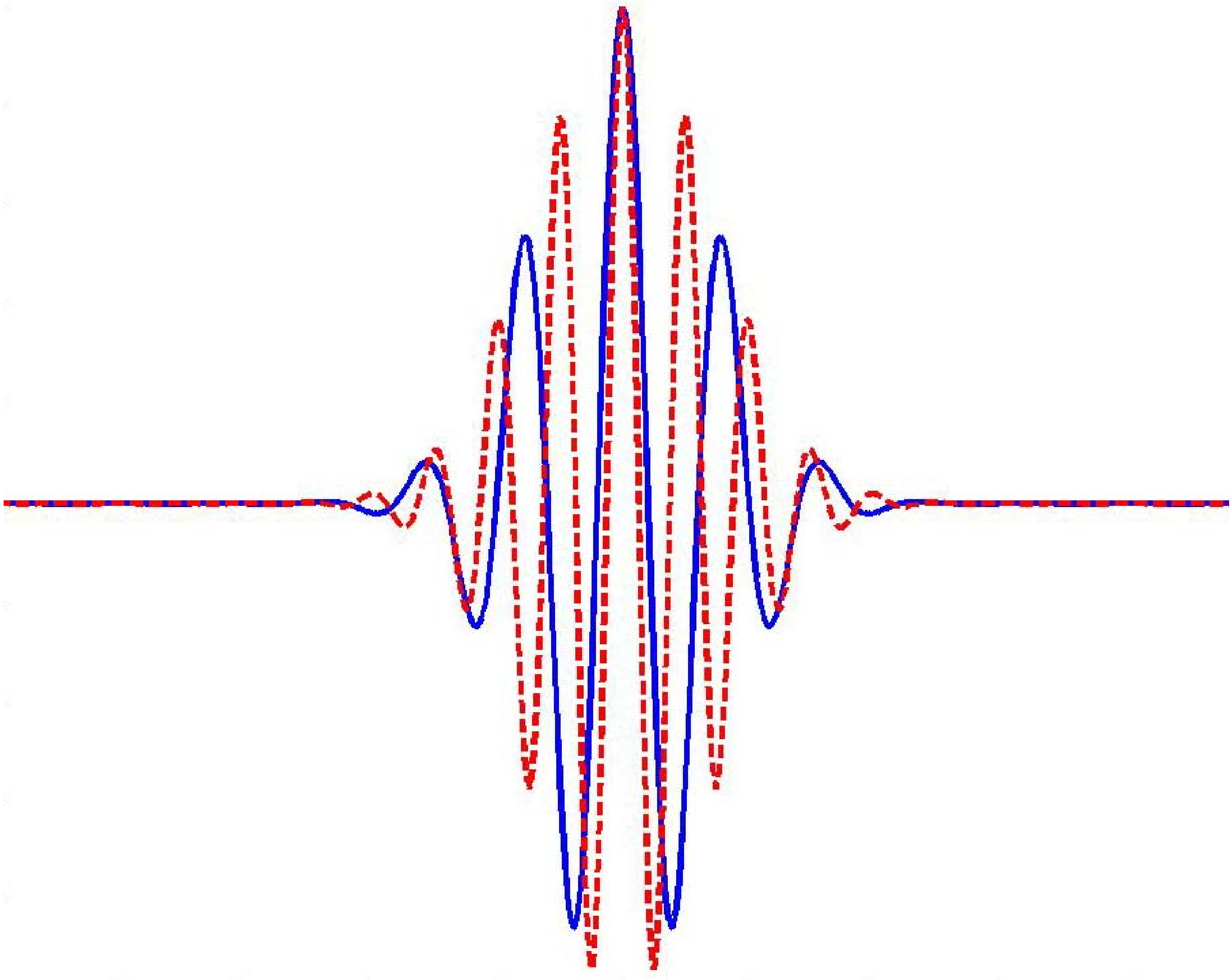
$$|S_t(f)|^2 = e^{-(\frac{f-f_0}{\sigma})^2}$$

$$|S_r(R,f)|^2 = |S_t(f)|^2 e^{-4\alpha R f} = e^{-(\frac{f-f_0}{\sigma})^2 - 4\alpha R f}$$

$$|S_r(R,f)|^2 = e^{-(\frac{f-f_1}{\sigma})^2} e^{-4\alpha R(f_0 - \sigma^2 \alpha R)}$$

$$f_1 = f_0 - 2\sigma^2 \alpha R.$$





Impulse Function

- Introduced in 1947 by Dirac
- Useful signal processing tool for
  - Sampling operations
  - Representing the transform of sinusoidal signals
- Impulse is a brief intense unit-area pulse that exists conceptually at a point

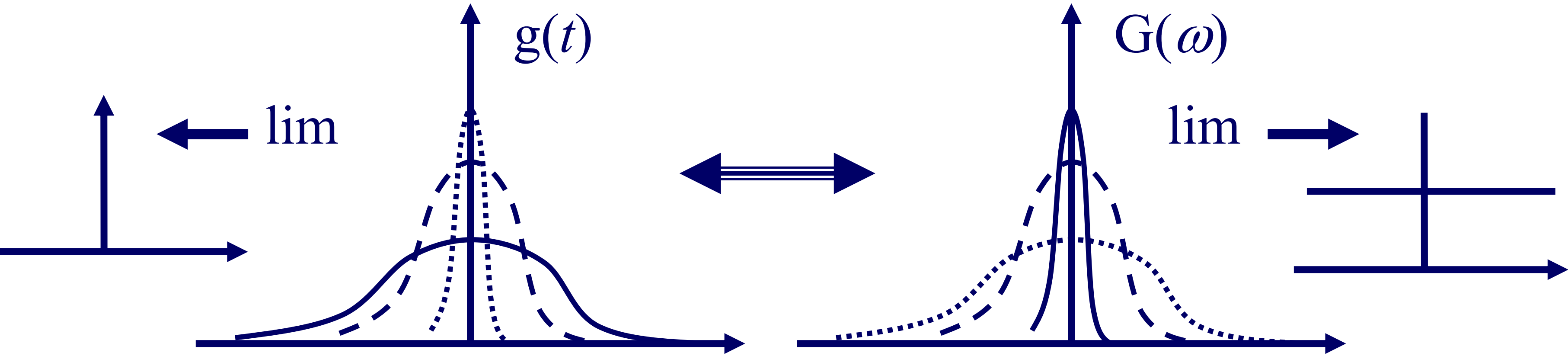
$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

Impulse Function

- Visualize as a limiting sequence of a window function, such as a Gaussian window

g(t)

$$\lim_{T \to 0} \int_{-\infty}^{\infty} T^{-1} g(t)\, dt = 1$$



Impulse Function

- Properties
  - Product

$$x(t)\delta(t) = x(0)\delta(t)$$

$$x(t)\delta(t-\tau) = x(\tau)\delta(t-\tau)$$

  - Convolution

$$x(t) * \delta(t) = x(t)$$

$$x(t) * \delta(t-\tau) = x(t-\tau)$$

  - Convolution with an impulse results in a shift operation

Continuous-Time System Response to Impulse Function

$$\text{Let } x(t) = \delta(t)$$

$$y(t) = \int_{-\infty}^{t} h(\tau)\delta(t-\tau)d\tau = h(t)$$

- Thus, *h(t)* has the interpretation as the impulse response of the continuous-time system (filter).

Fourier Transform and Properties

The Fourier Transform

- Forward Fourier transform (generally complex-valued)

$$FT\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- Reverse (backward) Fourier transform

$$FT^{-1}\{X(\omega)\} = x(t) = \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

- Base Fourier transform from which all other Fourier transform variations are derived.

The Fourier Transform

- Distinguishing terminology
  - aka Continuous-Time, Continuous-Frequency Fourier Transform
  - aka Continuous-Time, Fourier Transform (CTFT) where "transform" conveys the idea of continuous-frequency (Fourier "series" conveys discrete-frequency)
- Fourier transform amplitude and phase functions
  - Amplitude response, magnitude response, amplitude spectrum
  - Phase response, angle response, phase spectrum
  - Contrast these with temporal signal amplitude and phase functions

Even-Odd Signal Decomposition and Fourier Transform Properties

- Any function can be separated into even and odd components:

$$x(t) = x_{even}(t) + x_{odd}(t)$$

$$\text{where } x_{even}(t) = \frac{1}{2}\left[x(t) + x(-t)\right], x_{odd}(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$$

- Transform

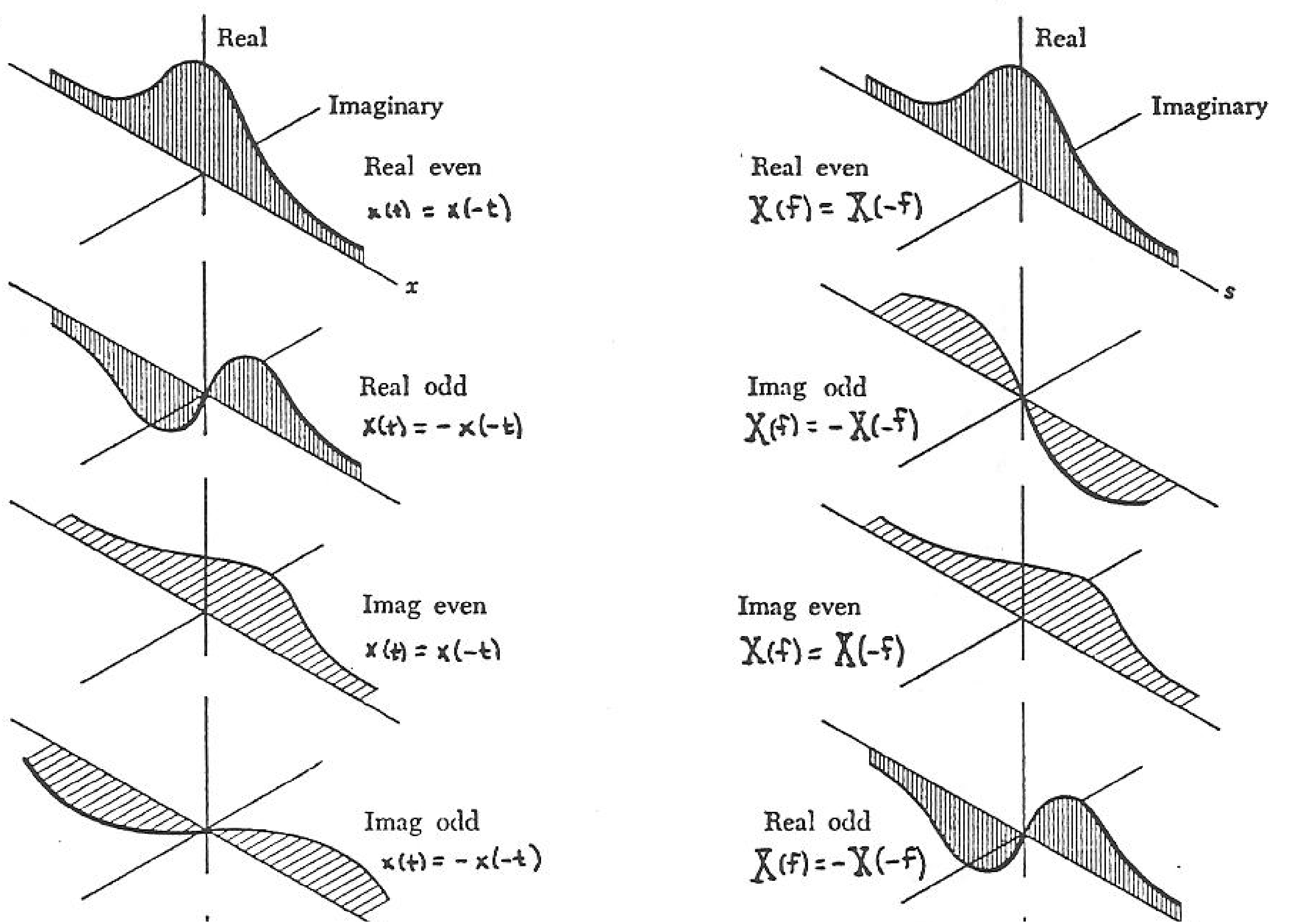
$$X(\omega) = X_{even}(\omega) + X_{odd}(\omega)$$

Even-Odd Signal Decomposition and Fourier Transform Properties

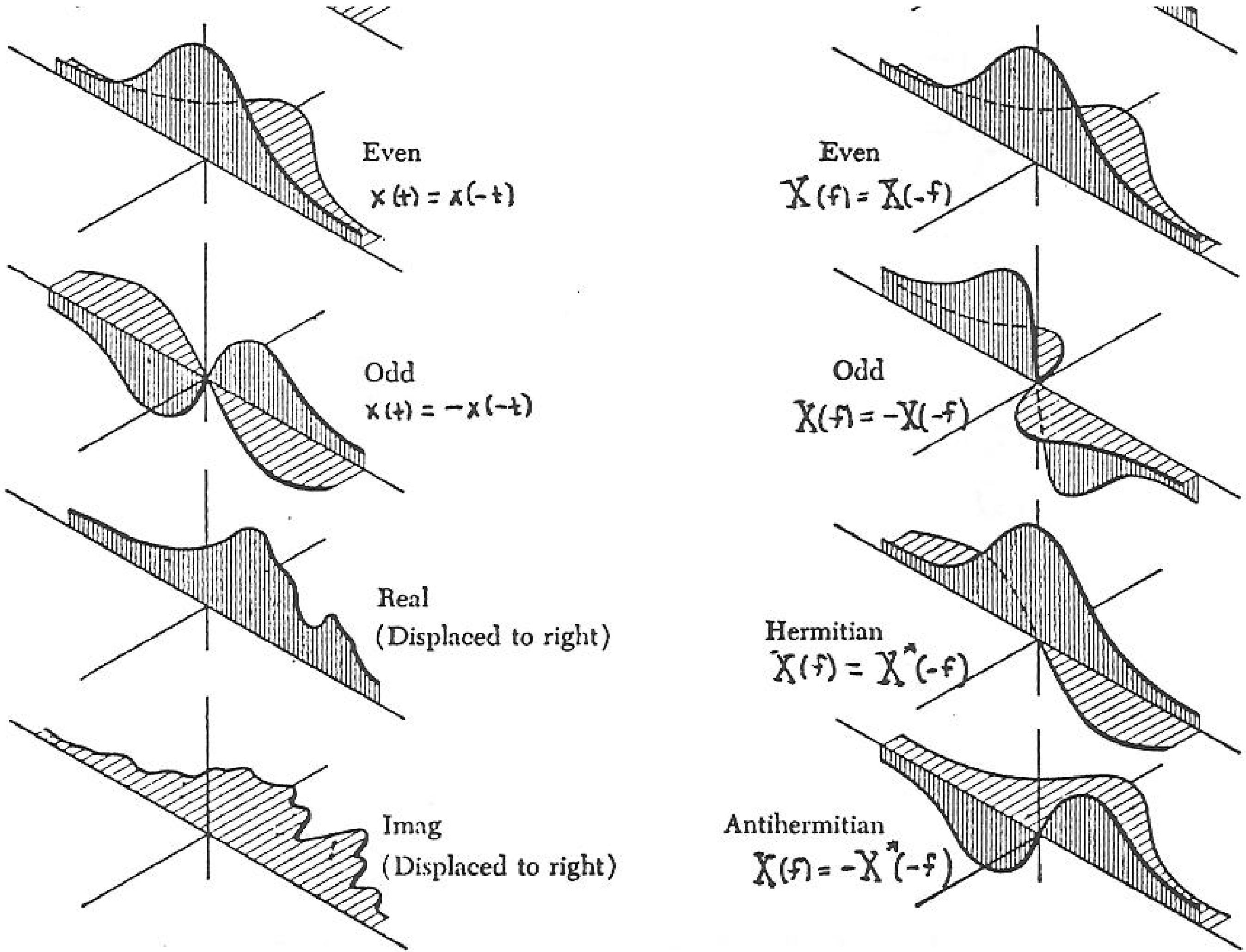
$$x(t) = \mathrm{Re}\{x_{even}(t)\} + j\,\mathrm{Im}\{x_{even}(t)\} + \mathrm{Re}\{x_{odd}(t)\} + j\,\mathrm{Im}\{x_{odd}(t)\}$$

$$X(\omega) = \mathrm{Re}\{X_{even}(\omega)\} + j\,\mathrm{Im}\{X_{even}(\omega)\} + \mathrm{Re}\{X_{odd}(\omega)\} + j\,\mathrm{Im}\{X_{odd}(\omega)\}$$

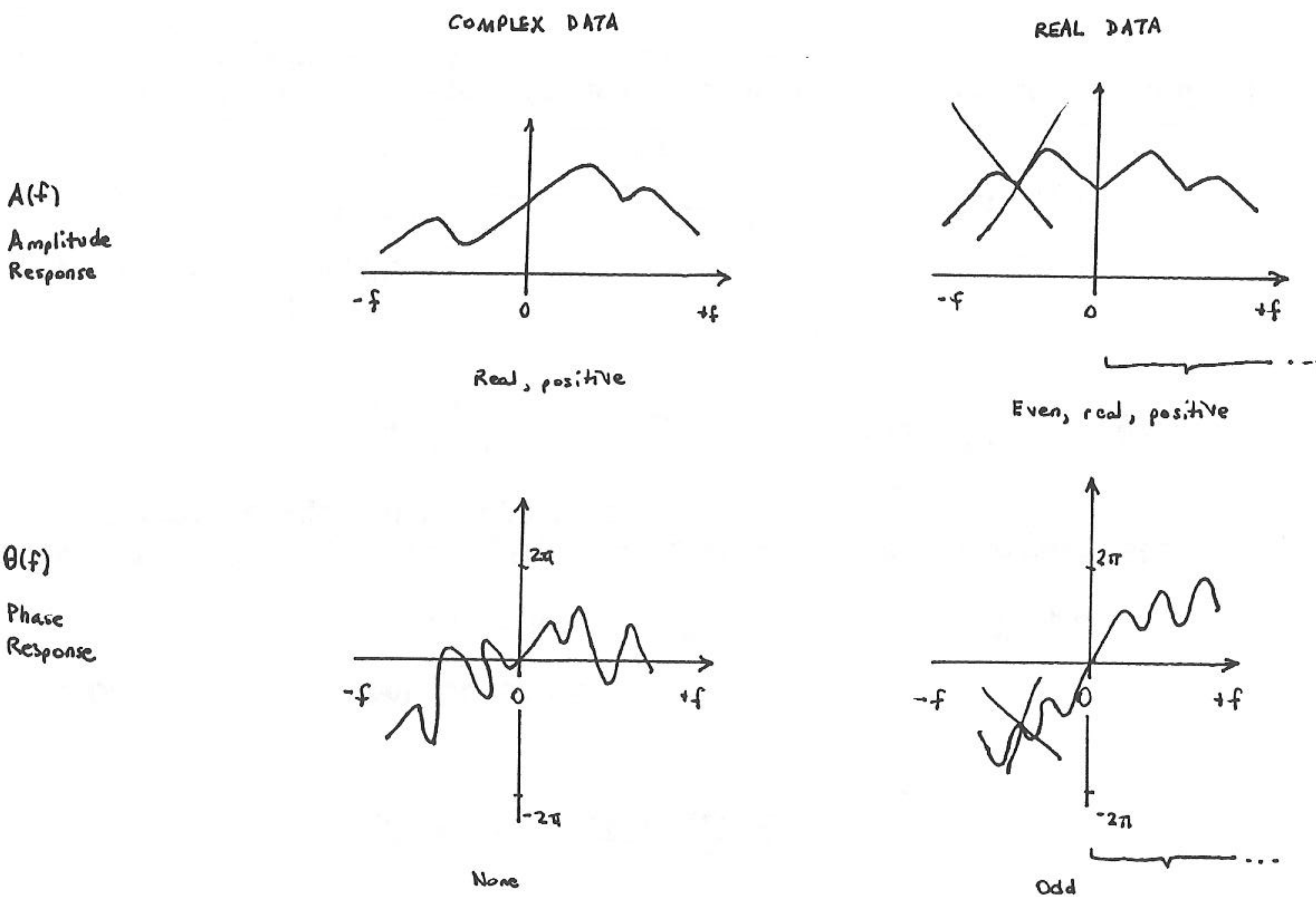
Symmetry Properties of a Signal and its Fourier Transform



Symmetry Properties of a Signal and its Fourier Transform



Visualizing the System Response



The Logarithmic Scale

- Definition of decibel (dB)

$$dB = 10\log_{10}(P / P_{ref})$$

- If $P = V^2/R$

$$dB = 10\log_{10}(V^2 / V_{ref}^2) = 20\log_{10}(V / V_{ref})$$

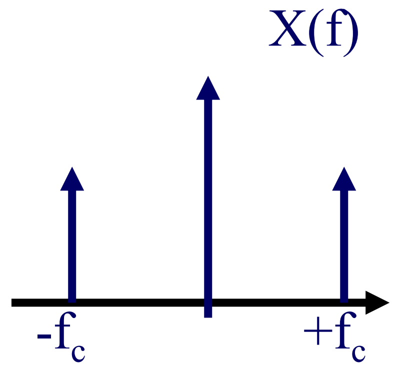
Summary of Key CTFT Properties and Functions

PROPERTY or FUNCTION	FUNCTION	TRANSFORM
Linearity	$ag(t) + bh(t)$	$aG(f) + bH(f)$
Time Shift	$h(t - t_0)$	$H(f)\exp(-j2\pi f t_0)$
Frequency Shift (Modulation)	$h(t)\exp(j2\pi f_0 t)$	$H(f - f_0)$
Scaling	$(1/\|\alpha\|)\, h(t/\alpha)$	$H(\alpha f)$
Temporal Convolution Theorem	$g(t) \otimes h(t)$	$G(f) \cdot H(f)$
Frequency Convolution Theorem	$g(t) \cdot h(t)$	$G(f) \otimes H(f)$
Window Function	$A\,\text{wind}(t/T_0)$	$2AT_0\,\text{sinc}(2T_0 f)$
Sinc Function	$2AF_0\,\text{sinc}(2F_0 t)$	$A\,\text{wind}(f/F_0)$
Impulse Function	$A\delta(t)$	A
Sampling (Replicating) Function	$Ш_T(t)$	$F\,Ш_F(f)$, $F = 1/T$

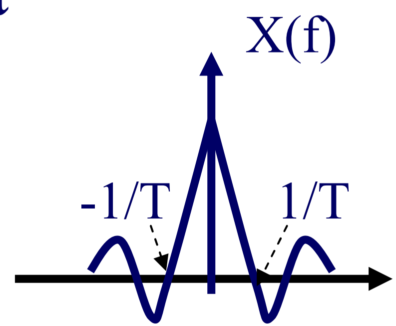
Special Signals and Their Transforms

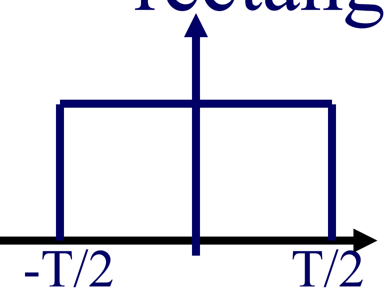
- Cosine signal

$$x(t) = \cos 2\pi f_c t = \frac{1}{2} e^{j2\pi f_c t} + \frac{1}{2} e^{-j2\pi f_c t}$$



- Time-domain window function (aka rectangular window)

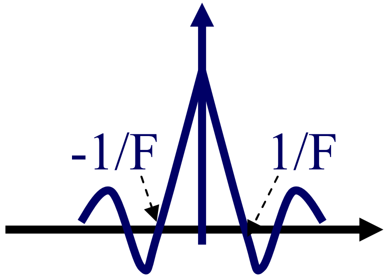




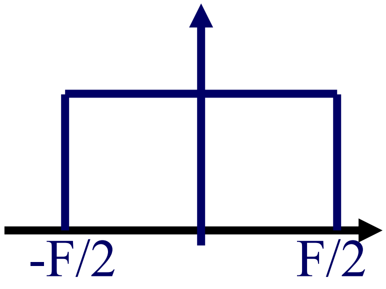
$$\Pi_T(t) = \begin{cases} 1, |t| < T/2 \\ 1/2, |t| = T/2 \\ 0, |t| > T/2 \end{cases} \Leftrightarrow 2T \sin c(2Tf), \sin c(t) = \frac{\sin \pi t}{\pi t}$$

Special Signals and Their Transforms

- Frequency-domain window function



$$2F\,\mathrm{sinc}(2Ft) \Leftrightarrow \Pi_F(f)$$



Sign Function

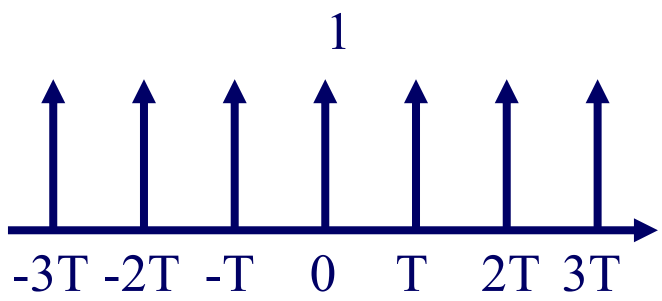
$$x(t) = \operatorname{sgn}(t) = \begin{cases} 1, t > 0 \\ 0, t = 0 \\ -1, t < 0 \end{cases} \Leftrightarrow X(f) = \frac{-j}{\pi f}$$

- Will be useful to develop the Hilbert transform

Impulse Train

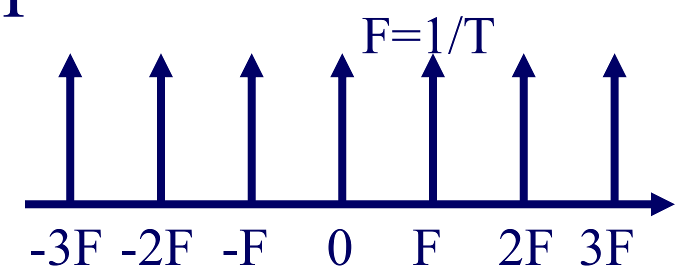
- Infinite periodic sequence of impulse functions spaced T seconds apart

$$\sum_{n=-\infty}^{\infty} \delta(t - nT)$$



- Transform is another impulse train

$$\frac{1}{T}\sum_{m=-\infty}^{\infty} \delta(f - \frac{m}{T})$$



Impulse Train

- Properties:
  - Product: In this case, the impulse train is called a sampling function.

$$\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

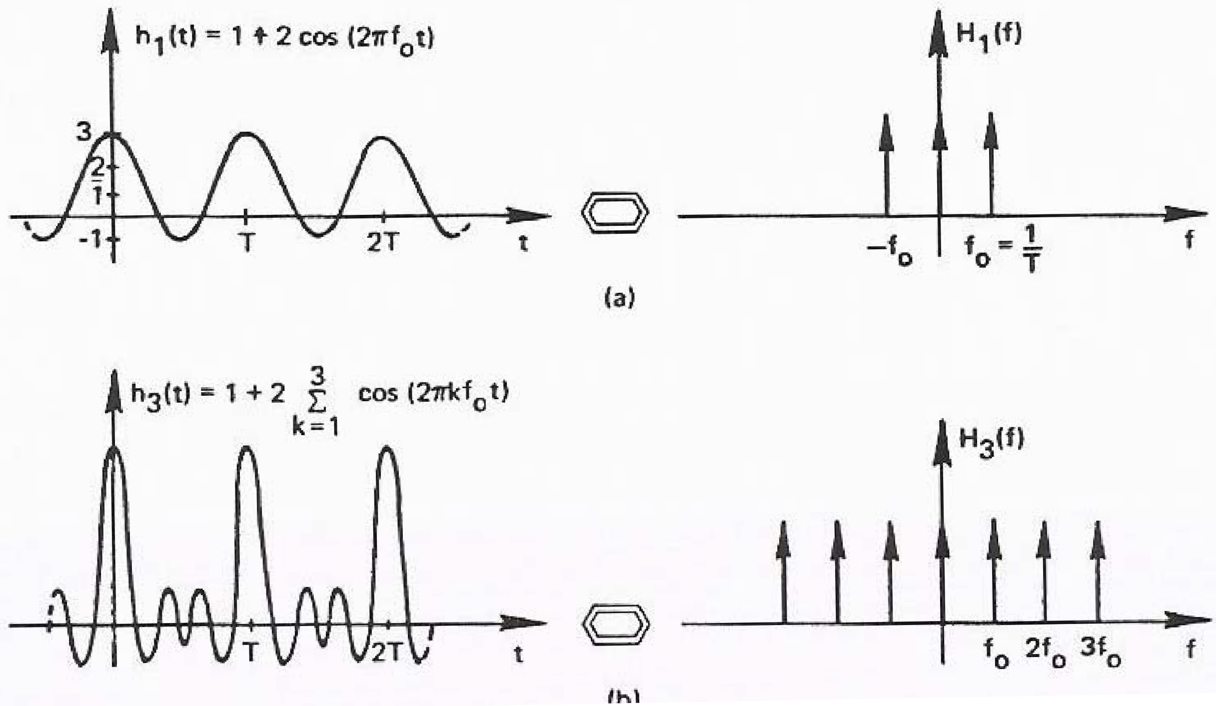
  - Convolution: In this case, the impulse train is called a replicating function.

$$\sum_{m=-\infty}^{\infty} x(t-mT)$$

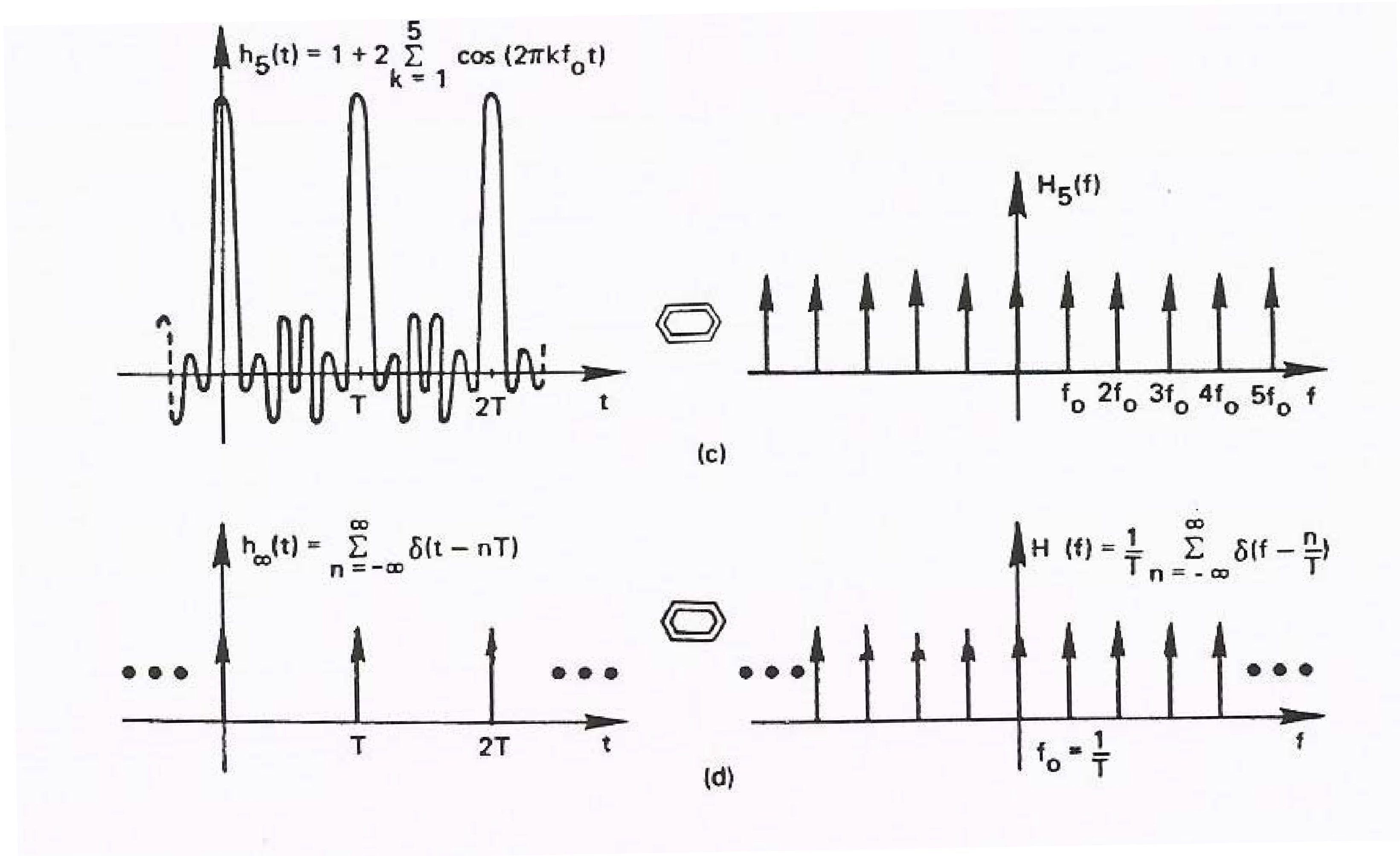
Graphical Illustration of Sequence of Impulse Functions

$$h(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \Longleftrightarrow H(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \qquad (2.44)$$

A graphical development of this Fourier transform pair is illustrated in Fig. 2.11.



Graphical Illustration of Sequence of Impulse Functions



Energy Preservation Between Domains

- Parseval-Rayleigh theorem

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$$

- Energy theorem (let $x(t)=y(t)$)

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df = energy$$

- Energy spectral density

$$|X(f)|^2$$

Matched Filter

- Objective: Determine system (filter) response $h(t)$ that maximizes the output energy of the system responses for the given input signal (assuming max is reached by time $t=t_0$).

$$y(t_0) = \int_{-\infty}^{t_0} x(t)h(t_0 - t)dt = FT^{-1}\{X(f)H(f)\}$$

- Based on Schwarz inequality

$$H(f) = cX^*(f) \text{ or } h(t) = cx^*(t_0 - t)$$

$$|y(t_0)|^2 = E \cdot \int_{-\infty}^{\infty} |H(f)|^2 df$$

Matched Filter

- Resulting operation is an autocorrelation.

$$y(t) = x(t) * x^*(-t) =$$

$$\int_{-\infty}^{t} x(\tau) x^*(\tau + t) d\tau = FT^{-1}\left\{X(f) X^*(f)\right\} = FT^{-1}\left\{\left|X(f)\right|^2\right\}$$

Matched Filter and SNR

- Assume the noise input to the filter ($X_N(f)$) is uncorrelated with the filter and has frequency independent distribution as a function of frequency. We have

$$\left\langle \left| X_N(f) \right|^2 \right\rangle \equiv N_0$$

- The output noise power becomes

$$\sigma^2 = \left\langle \int_{-\infty}^{\infty} \left| X_N(f) H(f) \right|^2 df \right\rangle = N_0 \int_{-\infty}^{\infty} \left| H(f) \right|^2 df = N_0 \int_{-\infty}^{\infty} h(t)^2 dt$$

Matched Filter and SNR

- When using the matched filter

$$SNR_{\max} = \frac{\int_{-\infty}^{\infty} x^2(t_0 - t)dt}{N_0} = \frac{\int_{-\infty}^{\infty} x^2(t)dt}{N_0} \equiv \frac{E}{N_0}$$

- The maximum signal-to-noise ratio is determined only by its total energy *E*, not by the detailed structure of the signal.

Time-Bandwidth Product

- Approach using the area metric

$$\alpha = \int_{-\infty}^{\infty} x(t)dt / x(0)$$

$$\beta = \int_{-\infty}^{\infty} X(f)df / X(0)$$

$$\alpha \cdot \beta = 1$$

- Rule of thumb: bandwidth of the pulsed signal is roughly the reciprocal of the signal's time duration.

Time-Bandwidth Product

- Approach using the variance metric

$$\alpha^2 = 4\pi^2 \int_{-\infty}^{\infty} t^2 |x(t)|^2 \, dt / E$$

$$\beta^2 = 4\pi^2 \int_{-\infty}^{\infty} f^2 |X(f)|^2 \, df / E$$

$$\alpha \cdot \beta \geq \text{a constant}$$

- Equality when $x(t)$ is Gaussian.
- Other metrics may be required to handle special cases (e.g., bandpass signals with no content near DC).

Range and Velocity Accuracy in Doppler Estimation

- Let

$$x(t) = x_0(t) + n(t)$$

- With a matched filter, we have (the maximum occurs at $t=0$)

$$h(t) = x_0(-t)$$

$$y(t) = \int_{-\infty}^{\infty} x_0(\tau) x_0(\tau - t)\, d\tau + \int_{-\infty}^{\infty} n(\tau) x_0(\tau - t)\, d\tau \equiv y_0(t) + \int_{-\infty}^{\infty} n(\tau) x_0(\tau - t)\, d\tau$$

- With noise, the maximum may shift to Δt. Our goal is to derive $< \Delta t^2 >$

Range and Velocity Accuracy in Doppler Estimation

- Taylor expansion

$$y_0(\Delta t) = y_0(0) + \frac{\Delta t^2}{2} y_0''(0) + R$$

- We have

$$y_0^{\ 2}(\Delta t) = y_0^{\ 2}(0) + \Delta t^2 y_0''(0)\, E + R$$

$$y_0^2(\Delta t) - y_0^2(0) = -\beta^2 E^2 \Delta t^2$$

$$y_0''(t) = \int_{-\infty}^{\infty} x_0''(\tau - t)\, x_0(\tau)\, d\tau$$

$$y_0''(0) = \int_{-\infty}^{\infty} x_0''(\tau)\, x_0(\tau)\, d\tau = -4\pi^2 \int_{-\infty}^{\infty} f^2 X_0(f) X_0^*(f)\, df$$

$$\beta^2 \equiv \frac{4\pi^2 \int_{-\infty}^{\infty} f^2 X_0(f) X_0^*(f)\, df}{\int_{-\infty}^{\infty} X_0(f) X_0^*(f)\, df} = \frac{4\pi^2 \int_{-\infty}^{\infty} f^2 X_0(f) X_0^*(f)\, df}{E}$$

$$y_0''(0) = -\beta^2 E$$

Range and Velocity Accuracy in Doppler Estimation

- Define a noise signal

$$\left\langle \varepsilon^2 \right\rangle \equiv \left\langle y_0^2(0) - y_0^2(\Delta t) \right\rangle \equiv \left\langle \int_{-\infty}^{\infty} \left| n(\tau) x_0(\tau) \right|^2 d\tau \right\rangle$$

- We have

$$\left\langle \varepsilon^2 \right\rangle = N_0 E = \beta^2 E^2 \left\langle \Delta t^2 \right\rangle$$

$$\left\langle \Delta t^2 \right\rangle = \frac{1}{\beta^2 (E / N_0)}$$

Range and Velocity Accuracy in Doppler Estimation

- Similarly

$$\left\langle \Delta f^2 \right\rangle = \frac{1}{\alpha^2 (E / N_0)}$$

- Thus

$$\left[\left\langle \Delta t^2 \right\rangle \left\langle \Delta f^2 \right\rangle \right]^{1/2} = \frac{1}{\alpha\beta (E / N_0)}$$

- Gaussian signals give poorer simultaneous measurements of time and frequency than any other signal.

Essentially Time-Limited and Band-Limited Signals

- Signals cannot be simultaneously band-limited and time-limited.
- Important for discrete-time applications that signal be band-limited (for sampling) and also for pulses to be time-limited (finite memory)

Essentially Time-Limited and Band-Limited Signals

- Essentially time-limited

$$\left|X(f) - X_{TL}(f)\right|^2 = \left|X(f) - \int_{-T/2}^{T/2} x(t) e^{-j2\pi ft} dt\right|^2 \leq \varepsilon_f$$

- Essentially band-limited

$$\left|x(t) - x_{BL}(t)\right|^2 = \left|x(t) - \int_{-B/2}^{B/2} X(f) e^{j2\pi ft} df\right|^2 \leq \varepsilon_t$$

Analytic and Causal Signals

- Causal and analytic signals are dual scenarios that link I/Q components through a Hilbert transform
- Causal signal is a signal that is 0 over negative time.
- Analytic signal is a complex signal with a transform that is 0 over negative frequency; created from a real signal.

Analytic and Causal Signals

- Causal signal

$$x(t) = x_e(t) + x_o(t) = x_e(t)\left[1 + \operatorname{sgn}(t)\right]$$

$$X(f) = X_e(f) * \left[\delta(f) - j\frac{1}{\pi f}\right] = X_e(f) - jX_e(f) * \frac{1}{\pi f}$$

- Analytic signal

$$x_a(t) = x(t) * \left[\delta(t) - j\frac{1}{\pi t}\right] = x(t) - jx(t) * \frac{1}{\pi t}$$

$$X_a(f) = X(f)\left[1 + \operatorname{sgn}(f)\right]$$

Hilbert Transform

- Transforms time → time or frequency → frequency

$$HT\{x(t)\} = -x(t) * \frac{1}{\pi t} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{(\tau - t)} d\tau$$

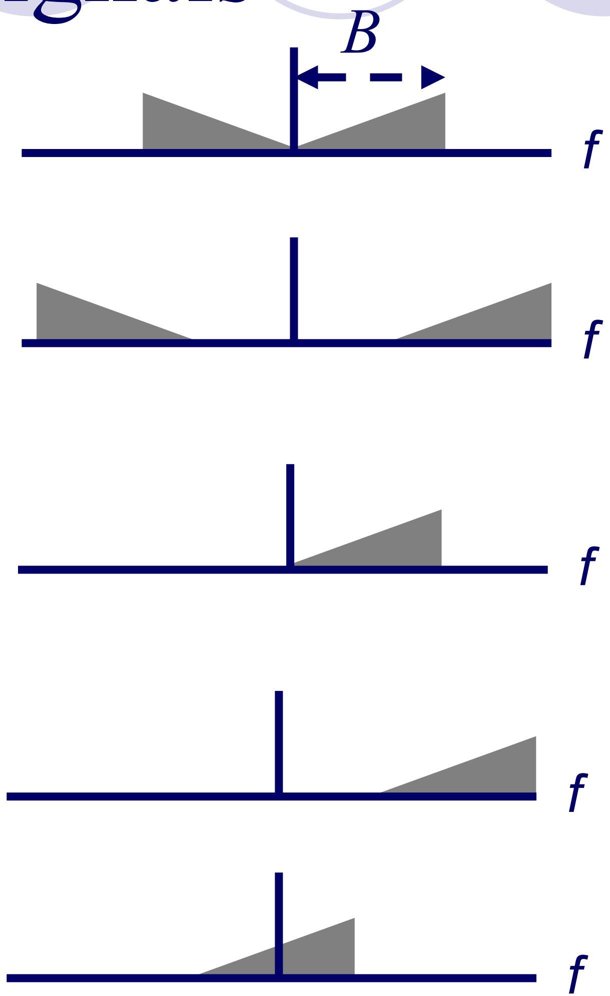
Sampling and Windowing Operations

Frequency Definitions

- Signal frequency
  - *F* (units of cycles per second, Hz)
- Sampling rate
  - $F=1/T$, *T* is sampling interval in seconds (per sample)
  - Units of samples per second
- Fraction of sampling rate
  - $f/F=fT$, dimensionless ratio (or cycles per sample)
  - Bounded by +/- 0.5 (normalized frequency)

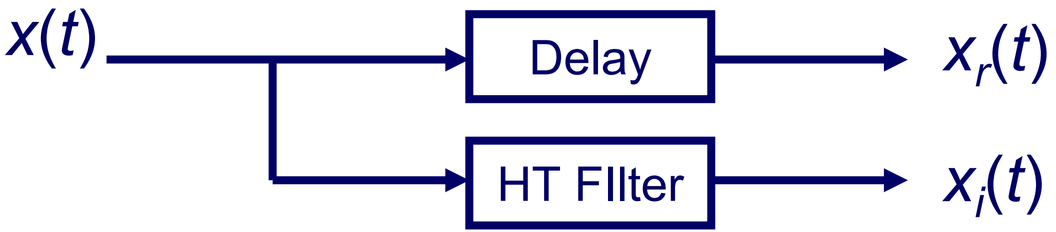
Band-Limited Transform Definitions for Continuous-Time Signals

- Baseband (lowpass) real signal
- Real bandpass signal
- Complex signal of one-sided baseband real signal
- Compex signal of one-sided bandpass real signal
- Baseband complex signal

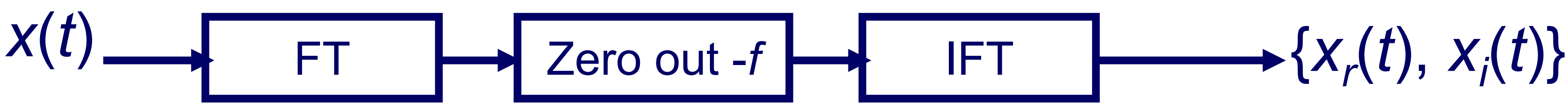


Creating One-Sided Complex Lowpass Signals from Real Lowpass Signals

- Analytic lowpass signal
- Time-domain approach

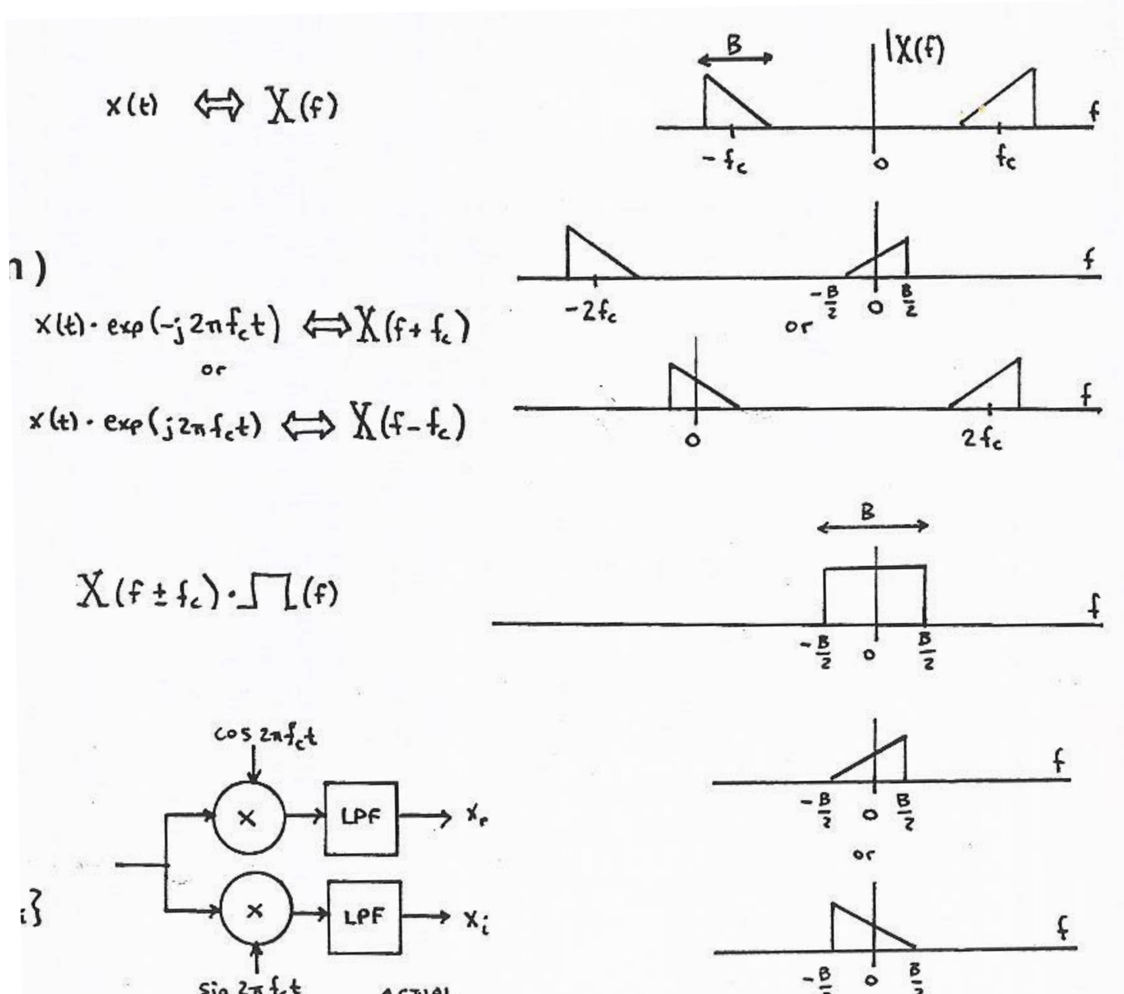


- Frequency-domain approach



Creating Baseband Complex Signals from Real Bandpass Signal

- Complex demodulation (quadratic demodulation)
- Original
- Band-shift
- Low pass filter
- Implementation

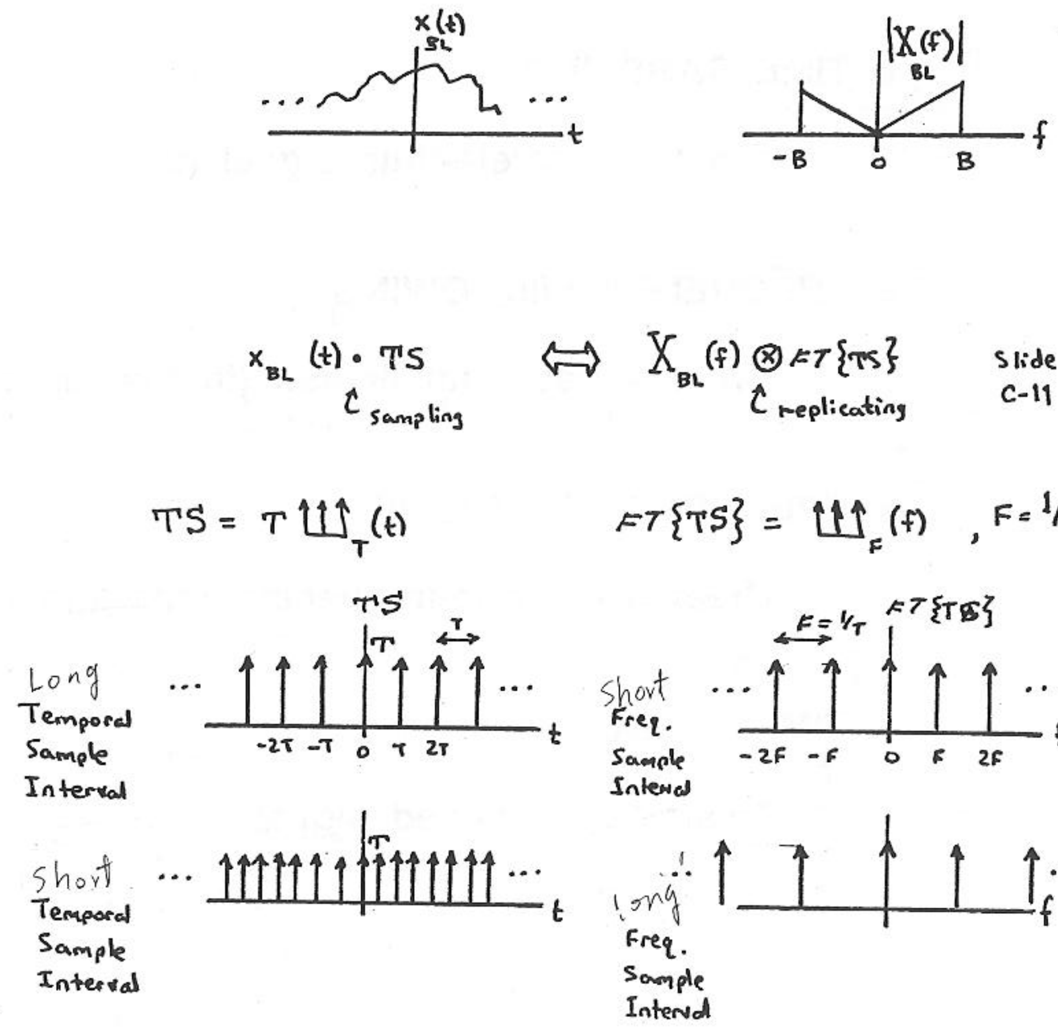


Four Basic Sampling and Windowing Operations Linking CT and DT Signals

- Time sampling:
  - Creates discrete-time signal (i.e., a time series)
- Frequency windowing:
  - Creates frequency-limited (i.e., band-limited) signal
- Frequency sampling:
  - Creates discrete frequency transform (i.e., a Fourier series)
- Time windowing:
  - Creates time-limited signal

Time Sampling Operation

- Band-limited requirement
- Time sampling operation
- Graphical Depiction



Selecting the Sampling Rate

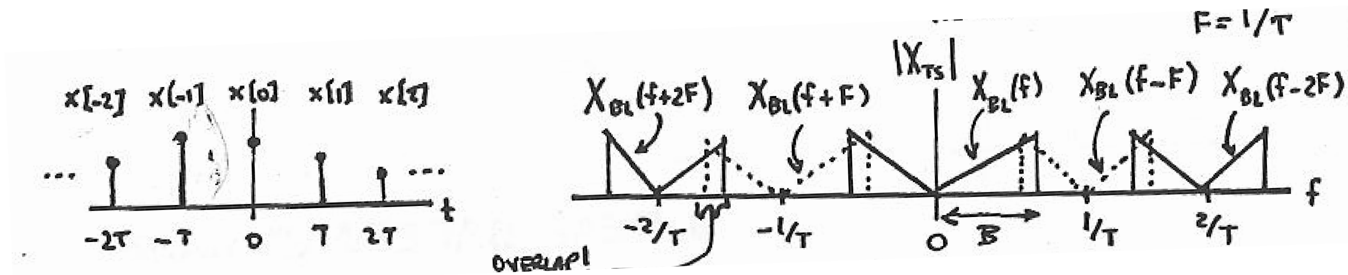
- Real baseband case
- Complex baseband case
- Real bandpass case

Sampling Real Baseband Signals

$$x_{TS}(t) = x_{BL}(t) \cdot TS = T\sum_{n=-\infty}^{\infty} x_{BL}(nT)\delta(t-nT) \Leftrightarrow \sum_{k=-\infty}^{\infty} X_{BL}(f-kF)$$

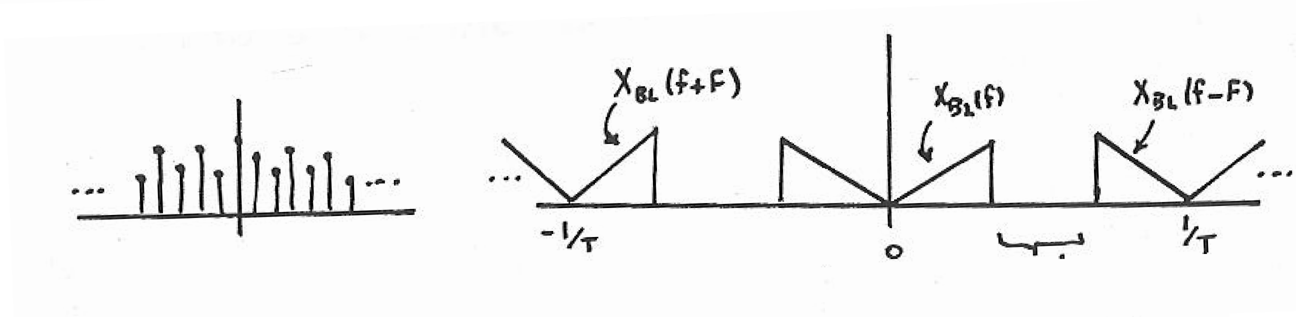
- Case 1

  $T > \frac{1}{2B}$



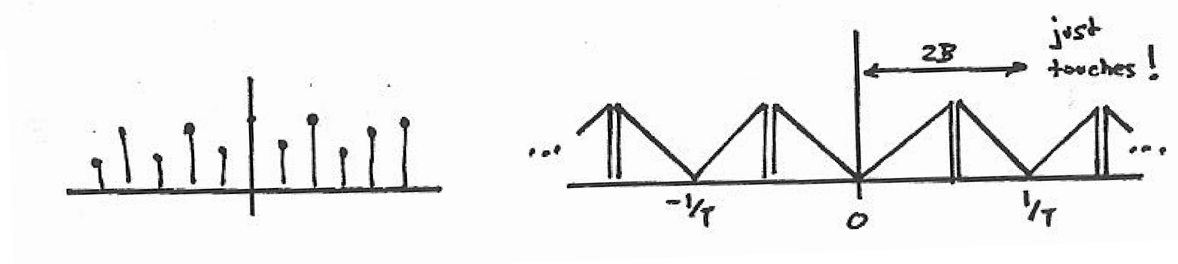
- Case 2

  $T < \frac{1}{2B}$



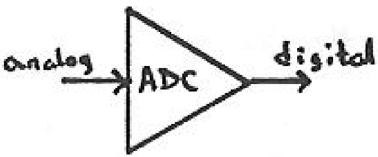
- Case 3

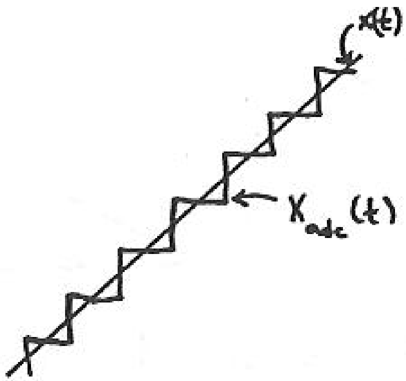
  $T = \frac{1}{2B}$



Strictly Band-Limited Signals Do Not Exist in Practice

- Some degree of aliasing cannot be avoided in actual hardware
- Analog anti-aliasing filters are imperfect
- Analog-to-digital converters introduce digitization noise

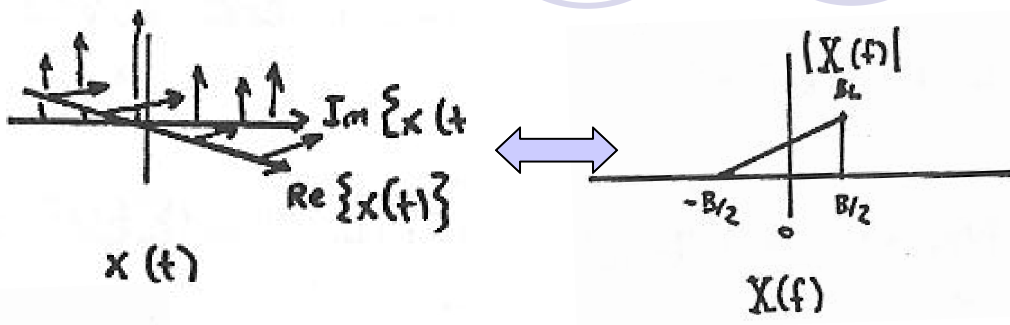




- Select sample rate based on bandwidth at which the signal is essentially band-limited.

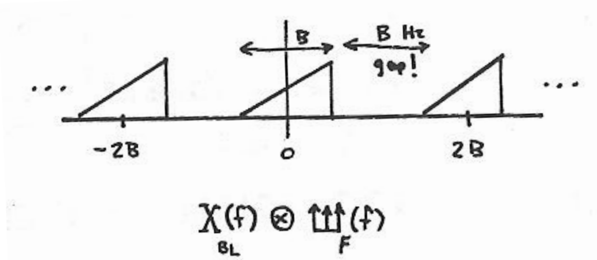
Sampling Complex Baseband Signals

- Signal and transform



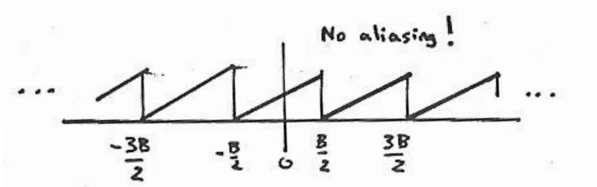
- Case 1

$$T = \frac{1}{2B}$$



- Case 2

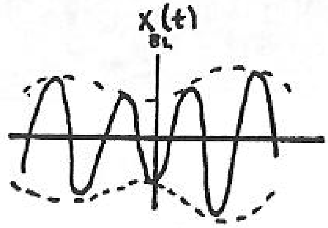
$$T = \frac{1}{B}$$

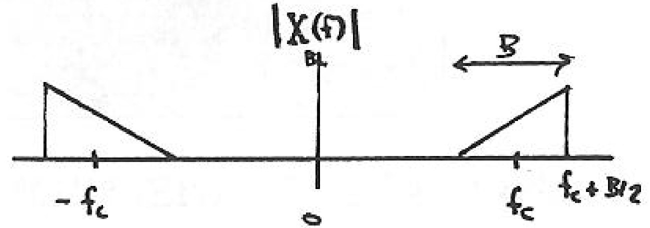


- Minimum sampling rate for complex baseband signal: F=B.

Sampling Real Bandpass Signals

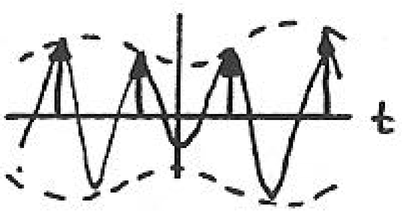
$$F = 2(f_c + \frac{B}{2})$$

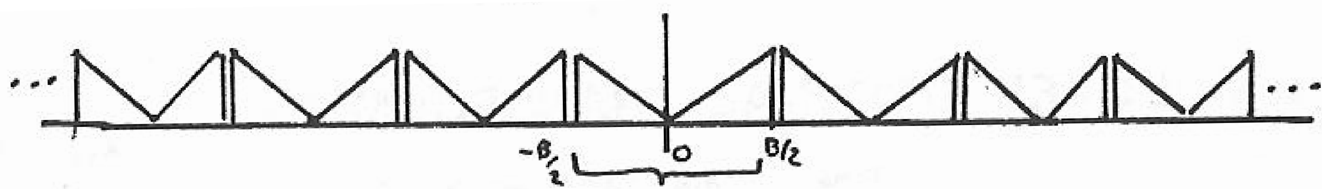




- Depending on relationship between F and B, can actually select sampling rate as low as $F=2B$ (baseband sampling theorem).
- Demodulation is free!!!

$$f_c = m2B + \frac{B}{2}; m \text{ is an integer}$$



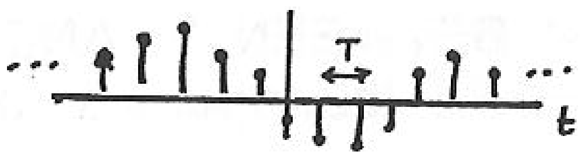


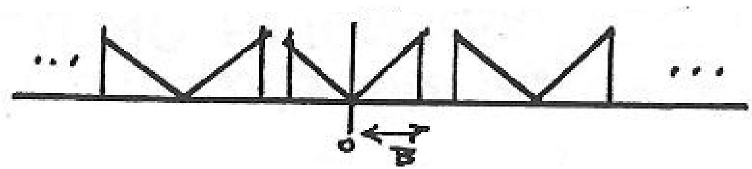
Sampling Real Bandpass Signals

- The above discussion is only valid for narrow band applications (fractional bandwidth is 40% when m=1).

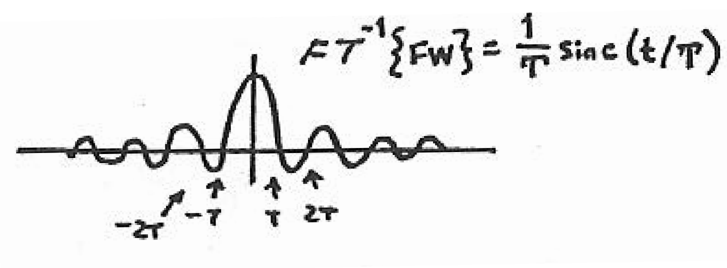
Frequency Windowing Operation

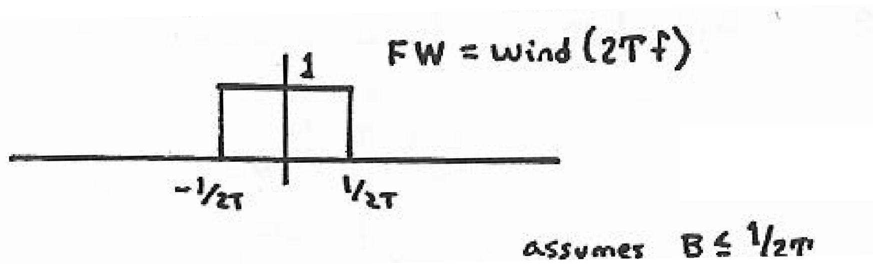
- Reconstruction of band-limited CT signal from DT signal
- Signal and transform





- Define frequency windowing operation (ideal low pass filter)





Frequency Windowing Operation

- Recover original transform by

$$x_{TS}(t) * FT^{-1}\{FW\} \Leftrightarrow X_{TS}(f) \cdot FW$$

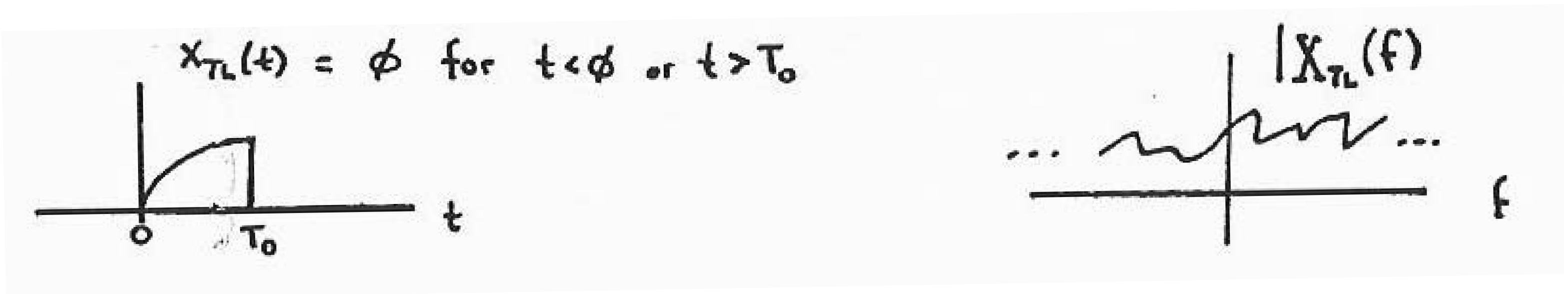
$$x_{BL}(t) = \sum_{n=-\infty}^{\infty} x_{BL}(nT) \sin c([t - nT]/T)$$

- Symbolic expression of the temporal sampling theorem

$$x_{BL}(t) = [x_{BL}(t) \cdot TS] * FT^{-1}\{FW\} \Leftrightarrow X_{BL}(f) = [X_{BL}(f) * FT\{TS\}] \cdot FW$$

Frequency Sampling Operation

- Dual to time sampling operation
- Assume continuous-time signal is time-limited, rather than band-limited

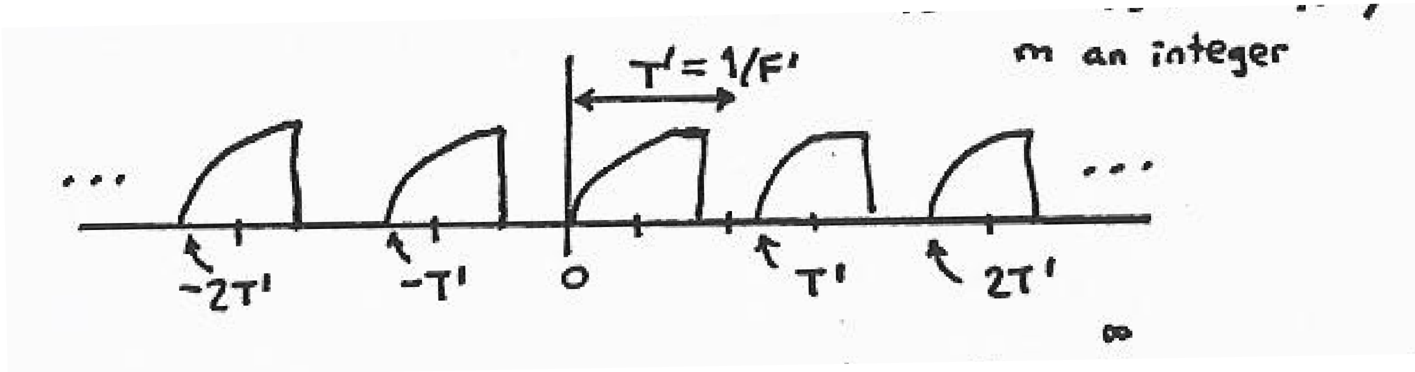


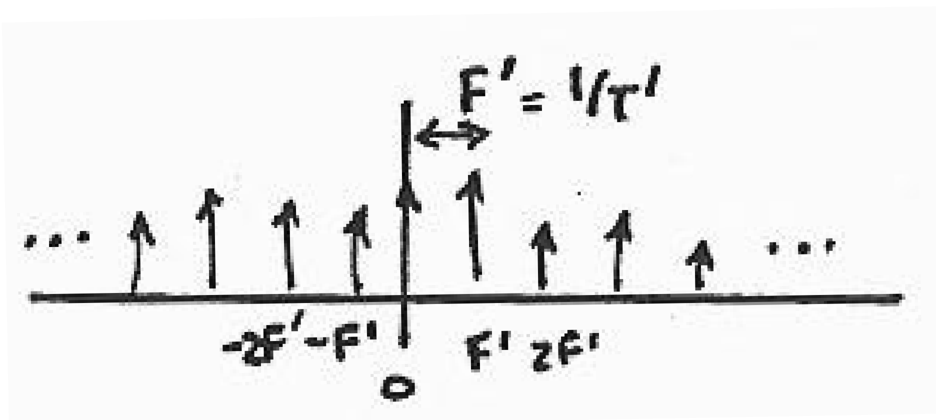
- Criteria to avoid temporal aliasing

$$F' \leq \frac{1}{T_0}$$

Frequency Sampling Operation

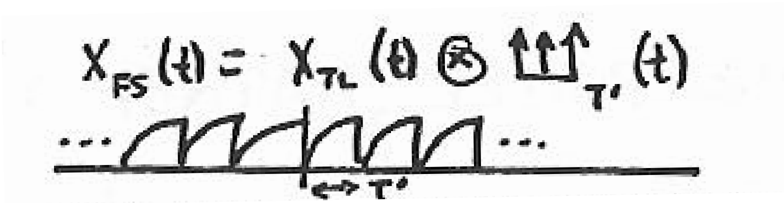
- Frequency sampling operation

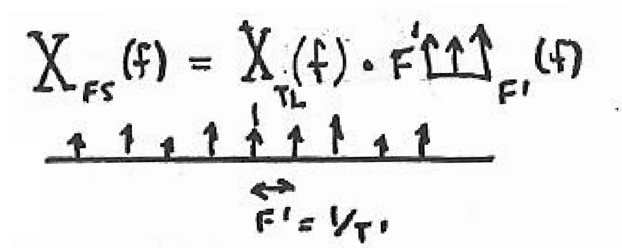




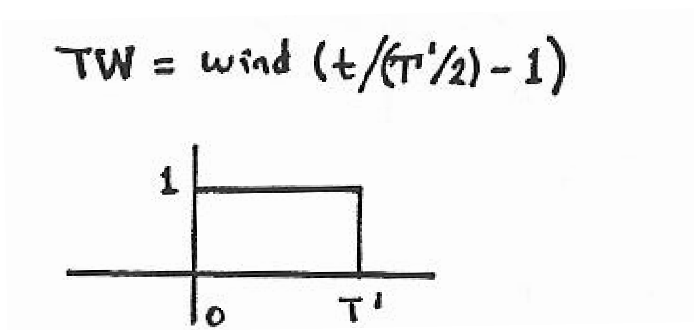
Time Windowing Operation

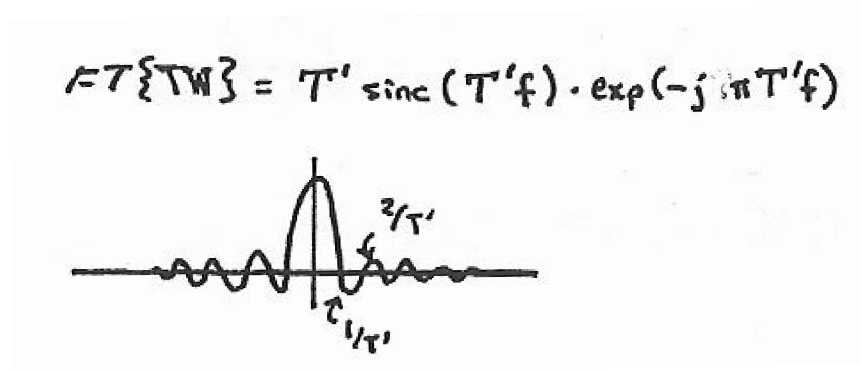
- Signal and transform





- Define one-sided time windowing operation





Time Windowing Operation

- Recovering original time signal

$$x_{FS}(t)\cdot TW \Leftrightarrow X_{FS}(f)*FT^{-1}\{TW\}$$

$$X_{TL}(t)=T'e^{-j\pi T'f}\sum_{k=-\infty}^{\infty}X_{FS}(kF')\sin c([f-kF']/F')$$

- Symbolic expression of frequency-domain sampling theorem

$$x_{TL}(t)=\left[x_{TL}(t)*FT^{-1}\{FS\}\right]\cdot TW \Leftrightarrow X_{TL}(f)=\left[X_{TL}(f)\cdot FS\right]*FT\{TW\}$$

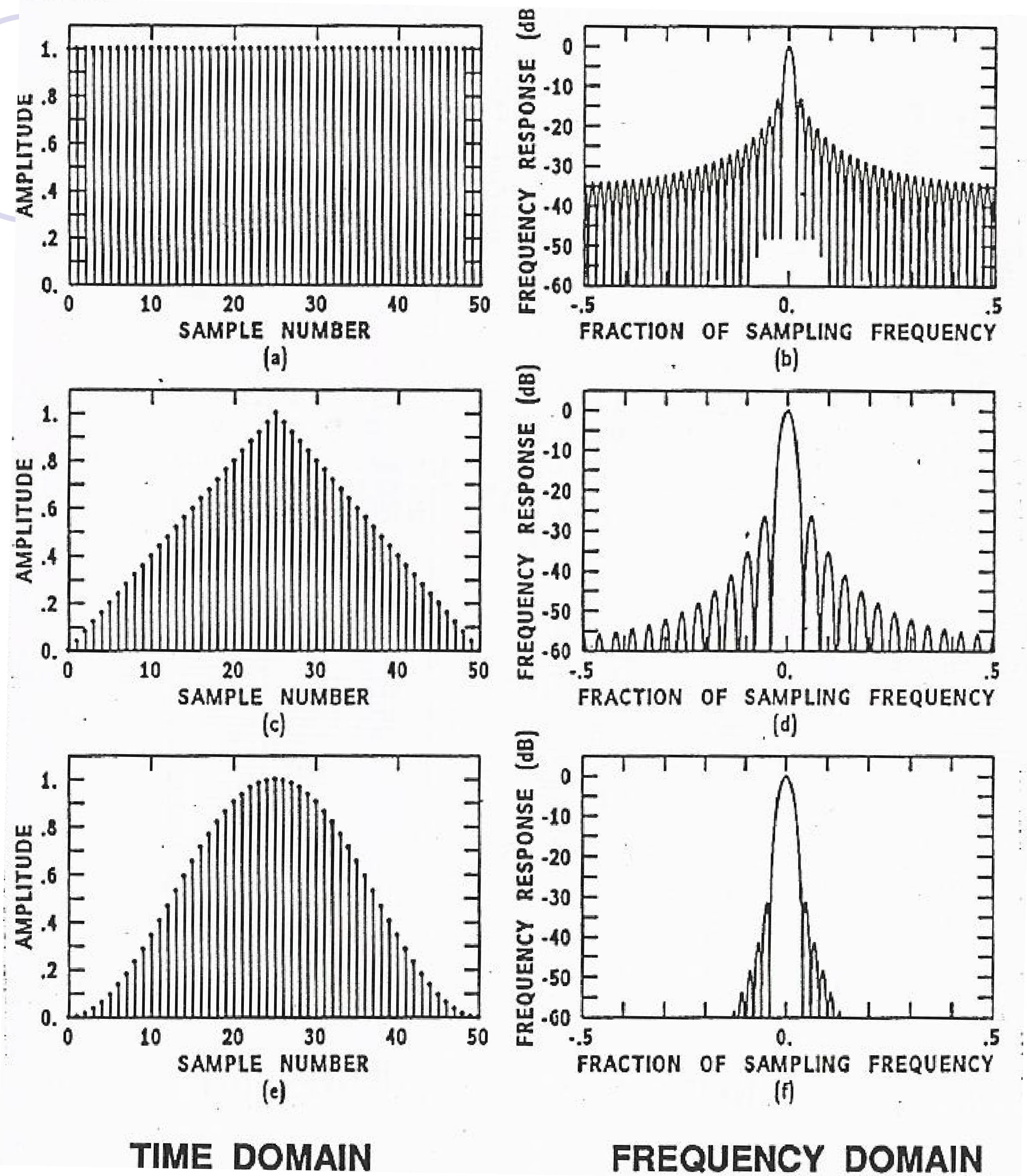
Uniform

(Rectangular)

Barlett

(Triangular)

Hann(ing)



Discrete-Time Signals and Transforms

Questions: How many Fourier Transforms?

- Fast Fourier Transform (FFT)
- Discrete-Time Fourier Transform (DTFT)
- Continuous-Time Fourier Series (CTFS)
- Continuous-Time Fourier Transform (CTFT)
- Discrete Fourier Transform (DFT)
- Fourier Series (FS)
- Discrete-Time Fourier Series (DTFS)

Answer: Just One!!!

- Fundamental: Continuous-Time Fourier Transform (CTFT)
- All other Fourier-Based transforms are derivable from the CTFT under specific signal conditions

Signal and Transform Relationships Using Both Time Sampling and Frequency Sampling

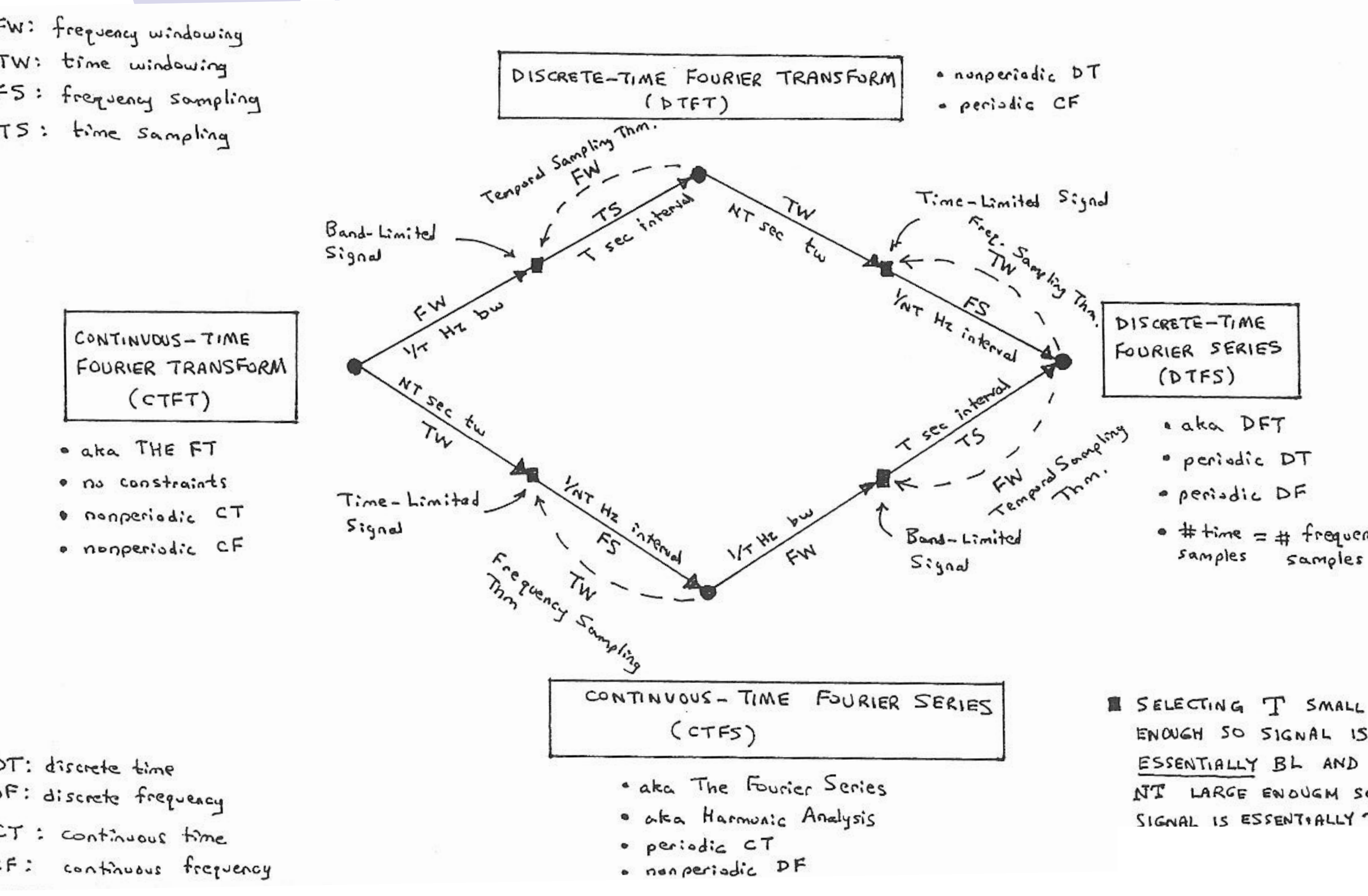
- General operations
  - Time limiting/Band limiting
  - Interpolation
  - Sampling
  - Replicating to create periodicity

Signal and Transform Relationships Using Both Time Sampling and Frequency Sampling

- Special case of four operations for scenario to derive DTFS (aka DFT)

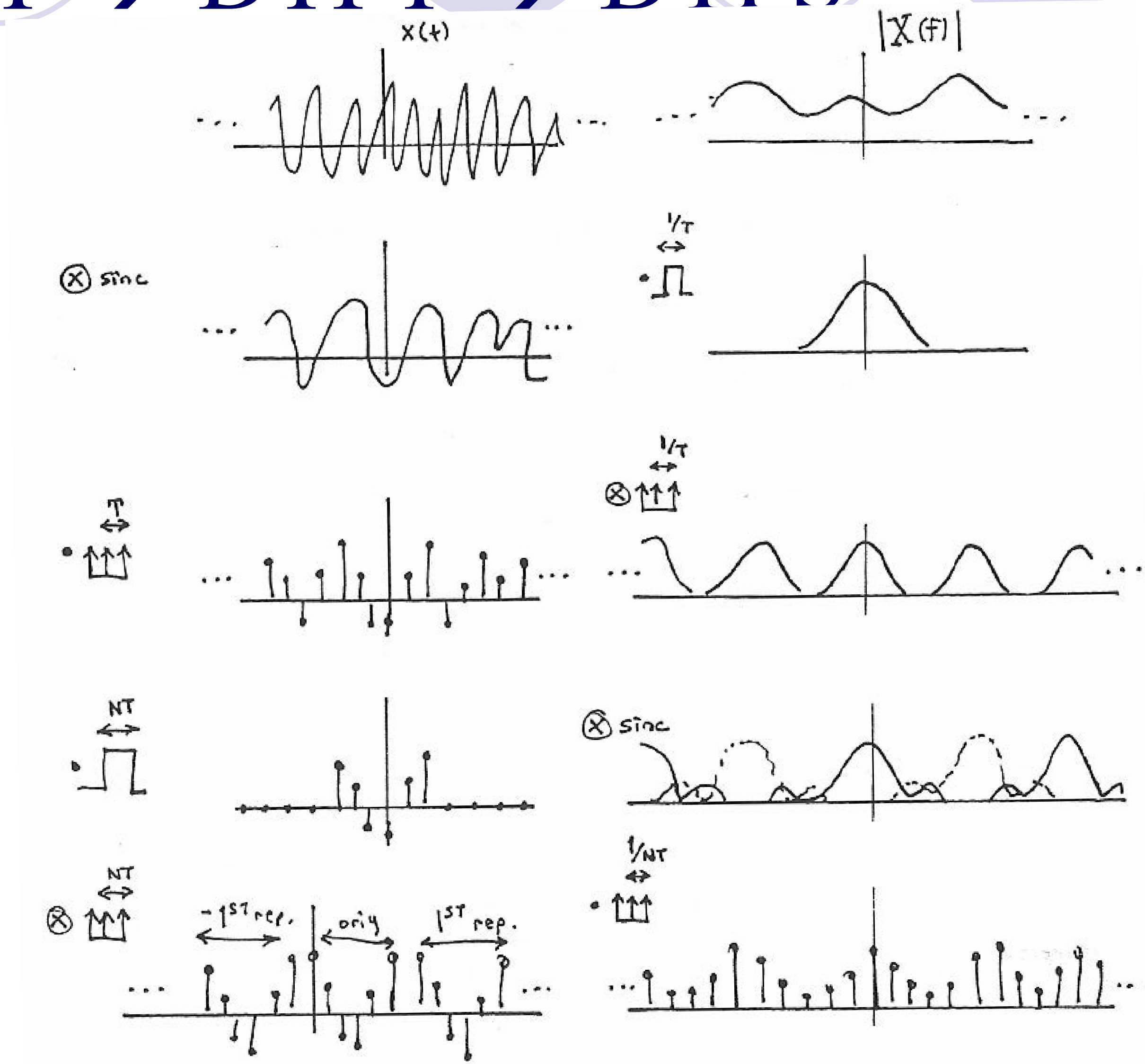
OPERATION	TIME FUNCTION	TRANSFORM FUNCTION
Time Windowing (NT-sec timewidth)	$\text{TW} = \text{wind}(2t/NT - 1)$	$\mathcal{F}\{\text{TW}\} = NT\text{sinc}(NTf) \cdot \exp(-j\pi NTf)$
Frequency Windowing ($1/T$-Hz bandwidth)	$\mathcal{F}^{-1}\{\text{FW}\} = \frac{1}{T}\text{sinc}(t/T)$	$\text{FW} = \text{wind}(2Tf)$
Time Sampling (T-sec intervals)	$\text{TS} = T\text{Ш}_T(t)$	$\mathcal{F}\{\text{TS}\} = \text{Ш}_{1/T}(f)$
Frequency Sampling ($1/NT$-Hz intervals)	$\mathcal{F}^{-1}\{\text{FS}\} = \text{Ш}_{NT}(t)$	$\text{FS} = \frac{1}{NT}\text{Ш}_{1/NT}(f)$

Four FTs through Sampling and Windowing



Graphical Representation of the Four Steps: CTFT → DTFT → DTFS

- Original
- FW
- TS
- TW
- FS



Continuous-Time Fourier Transform

- Transforms

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

- Energy preservation theorem

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

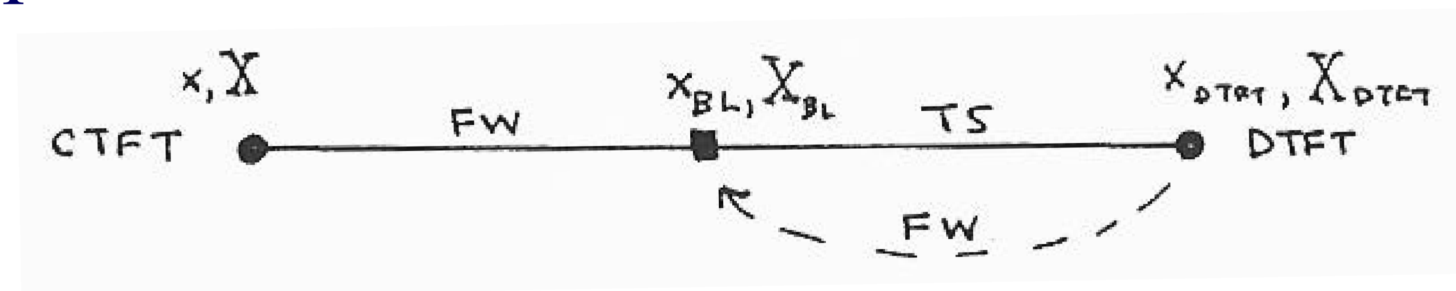
- Convolution theorem

$$x(t) \cdot y(t) \Leftrightarrow X(f) * Y(f)$$

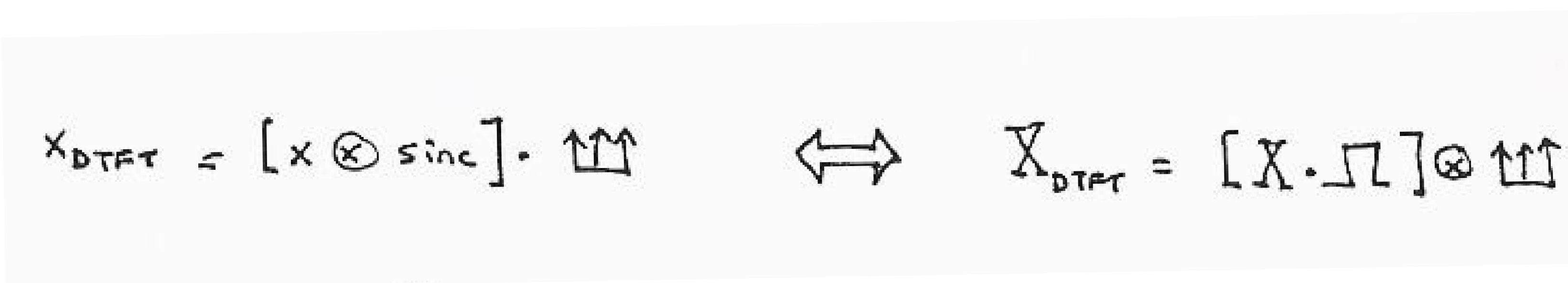
$$x(t) * y(t) \Leftrightarrow X(f) \cdot Y(f)$$

Discrete-Time Fourier Transform

- Operations



- Symbolic



Discrete-Time Fourier Transform

- Transforms

$$X_{DTFT}(f) = T\sum_{n=-\infty}^{\infty} x(nT)e^{-j2\pi fnT}$$

$$x_{DTFT}(nT) = \int_{-1/2T}^{1/2T} X_{DTFT}(f)e^{j2\pi fnT}df = x[n]$$

- Energy preservation theorem

$$T\sum_{n=-\infty}^{\infty} \left|x_{DTFT}(nT)\right|^2 = \int_{-1/2T}^{1/2T} \left|X_{DTFT}(f)\right|^2 df$$

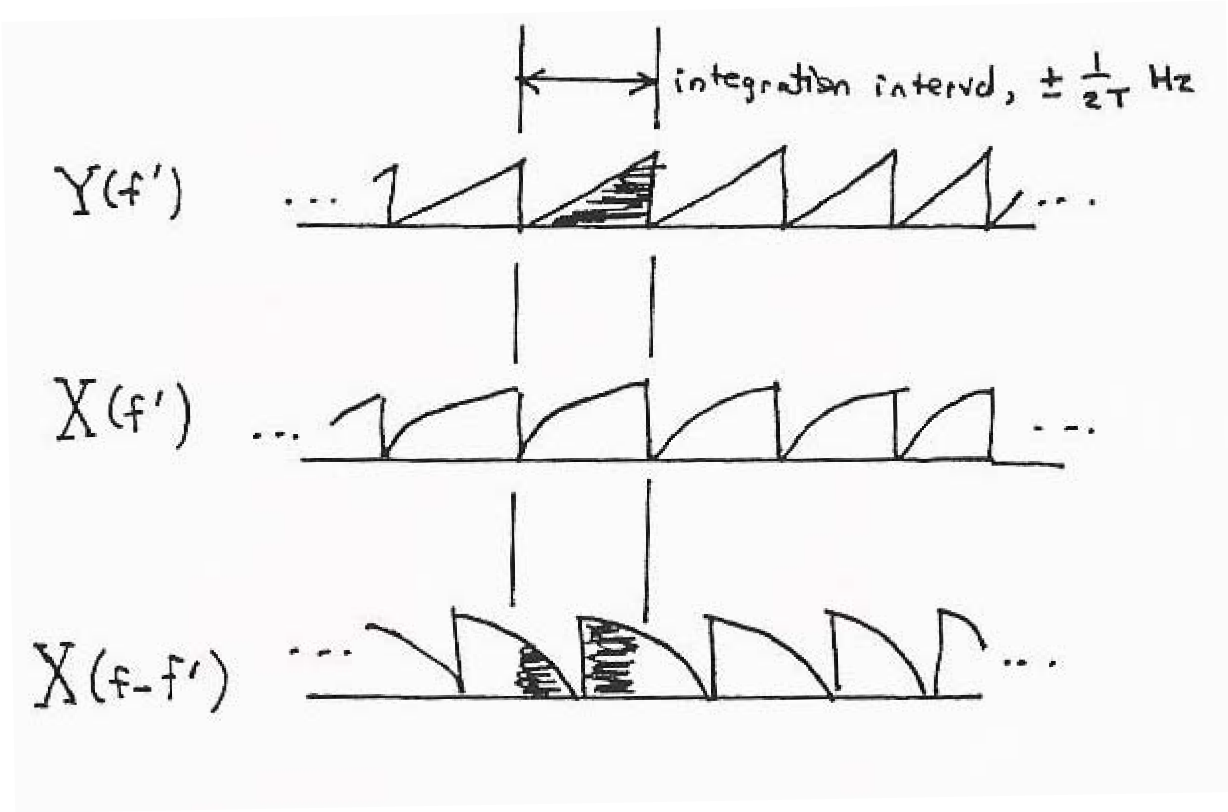
- Convolution theorem

$$x_{DTFT}(nT)\cdot y_{DTFT}(nT) \Leftrightarrow X_{DTFT}(f) * Y_{DTFT}(f)$$

$$x(nT) * y(nT) \Leftrightarrow X(f)\cdot Y(f)$$

Periodic Convolution

$$X_{DTFT}(f) \otimes Y_{DTFT}(f) = \int_{-1/2T}^{1/2T} X(f-f')Y(f')df'$$



Some Discrete-Time Fourier Transform Properties

- Transform of most interest in our case
- Can be computed at uniform frequency spacings for time-limited signals using the DTFS (aka DFT)
- Maintains CTFT even-odd properties and real-imaginary properties
- Time shift
- Frequency shift
- One-sided rectangular window