# Chapter 2:

# Acoustic Wave Propagation

# Outline

- Tools:
  - Newton’s second law
  - Hooke’s law
- Basics:
  - Strain, stress
  - Longitudinal, shear
  - Elastic moduli
- Acoustic wave propagation
  - Wave equations
  - Reflection, refraction
  - Impedance matching
- Applicatoin: Elasticity imaging

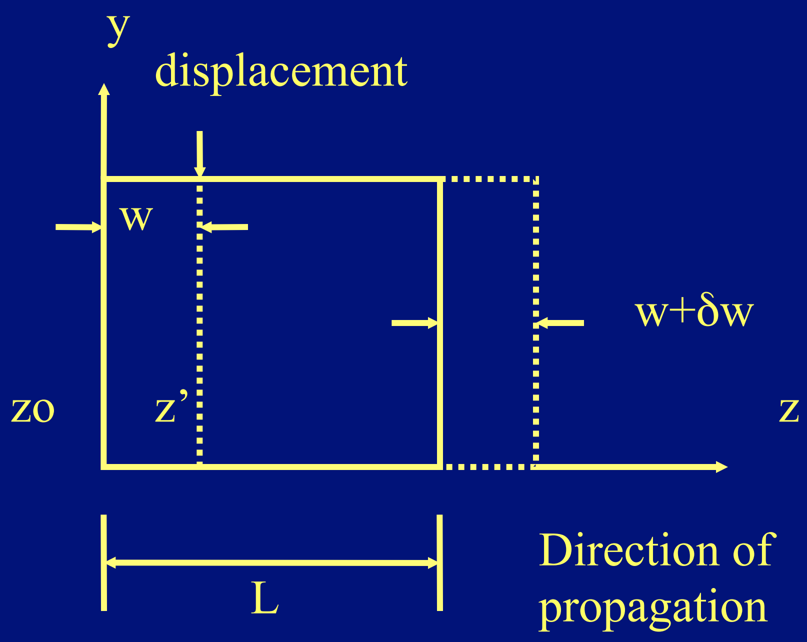
- Acoustics → Mechanical waves.
  - Newton's second law
  - Strain-displacement equations
  - Constitutive equations: Hooke's law
- Elastic medium:
  - Elastostatics: Stress-strain relationships
  - Elastodynamics: Wave equations

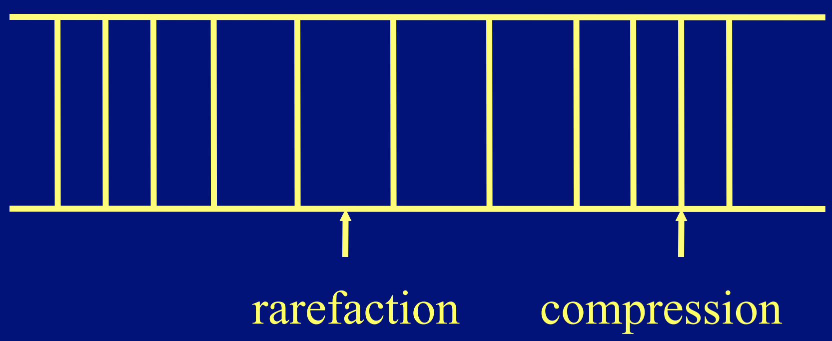
# Basics of Acoustic Waves

- A medium is required for a sound wave.
- Physical quantities to describe a sound wave: displacement, strain and pressure.
- Longitudinal (compressional) vs. shear (transverse).

# Basics of Acoustic Waves

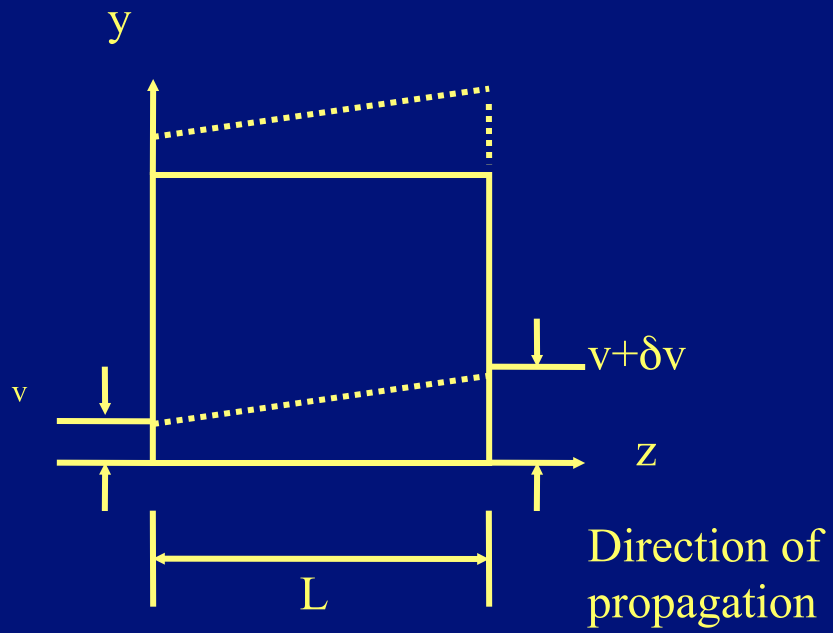
- Longitudinal Wave:

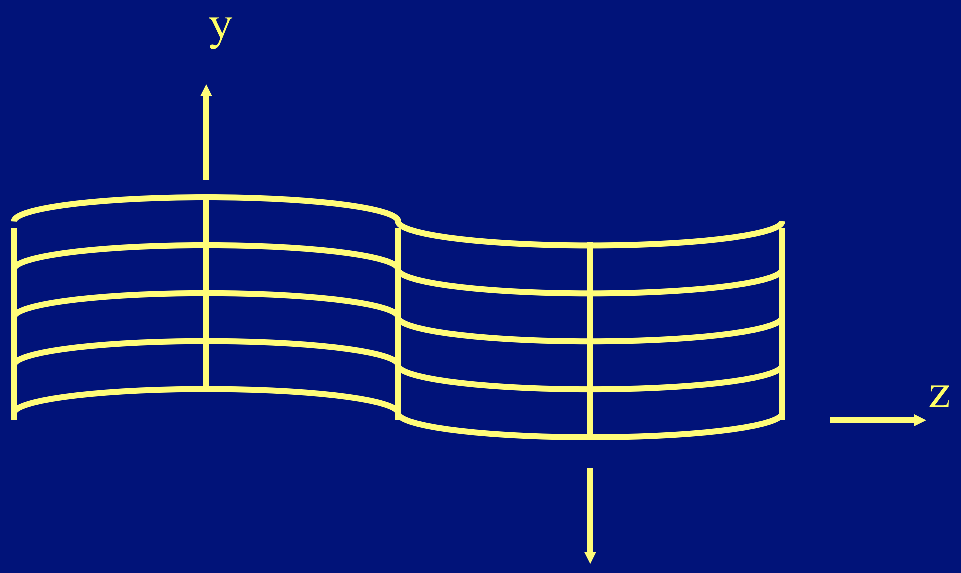


# Basics of Acoustic Waves

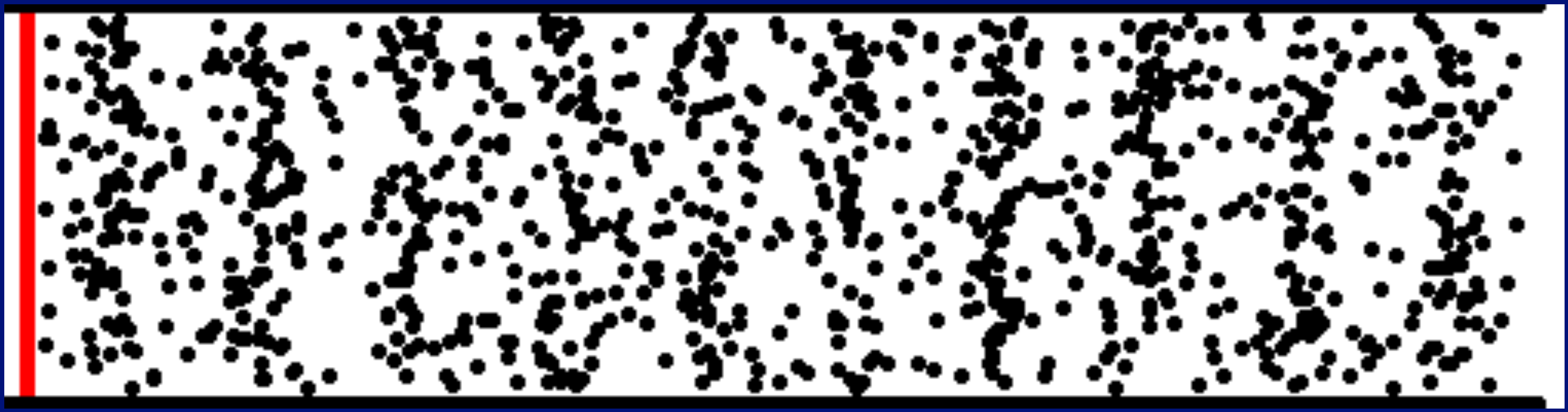
- Shear Wave (*No Volume Change*):



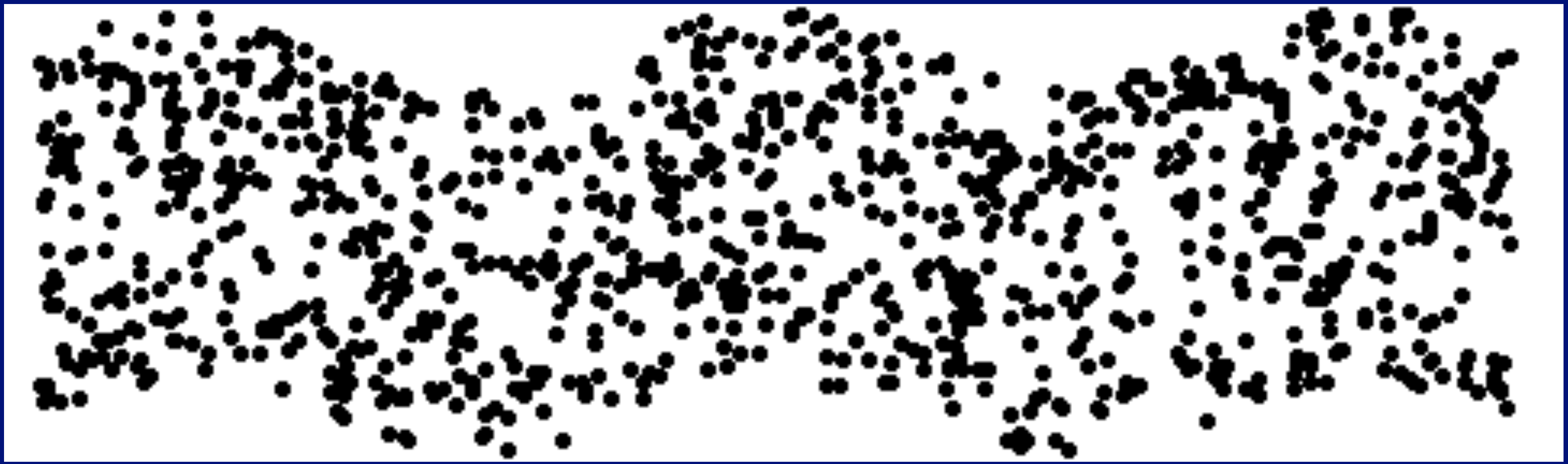
# Basics of Acoustic Waves

- Longitudinal Wave:



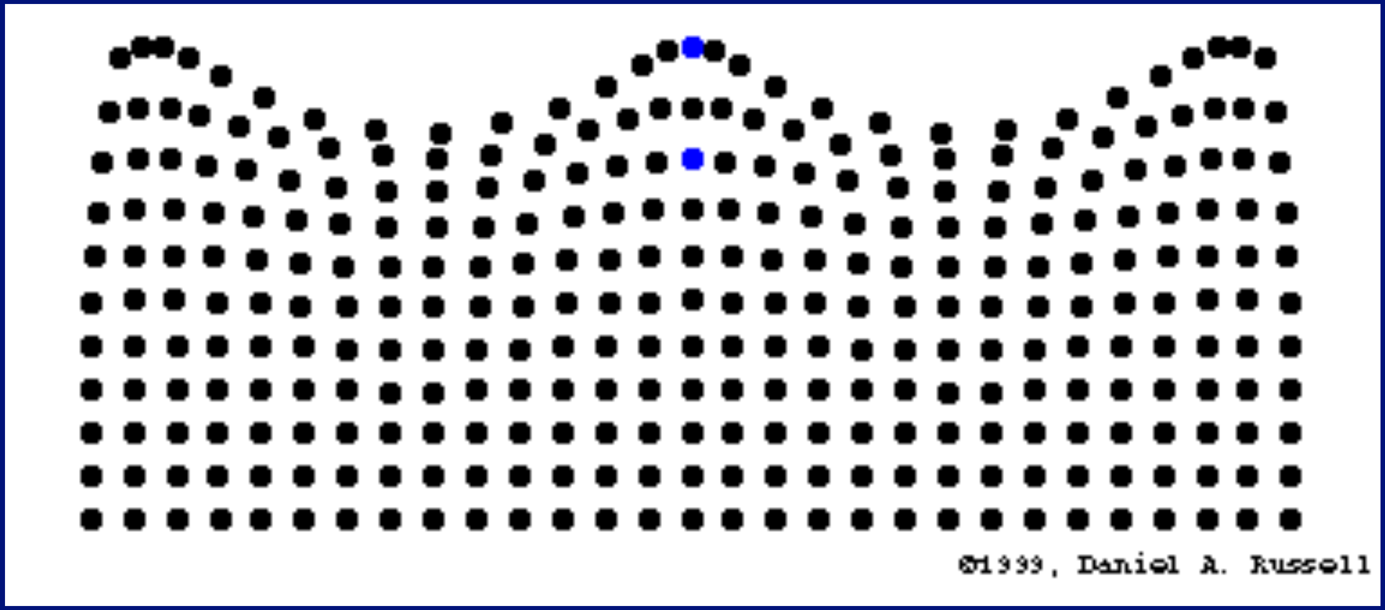
# Basics of Acoustic Waves

- Shear Wave:



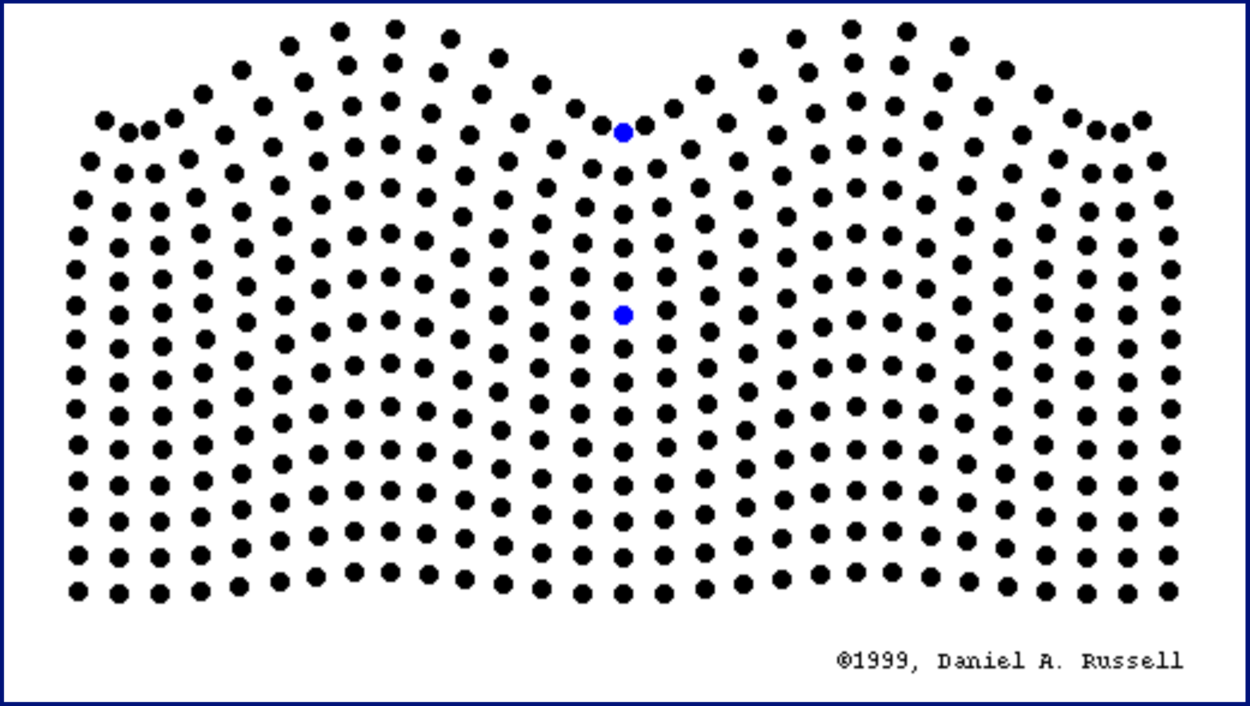
# Basics of Acoustic Waves

- Water Wave:



# Basics of Acoustic Waves

- Rayleigh Wave:



# Displacement and Strain

- Displacement: movement of a particular point.
- Strain:
  - Displacement variations as a function of position.
  - Fractional change in length.
  - Deformation.
  - Can be extended to volume change.

# Displacement and Strain

- Compressional strain:

$$\delta w = \frac{\partial w}{\partial z} L \equiv SL$$

$$S \equiv \frac{\partial w}{\partial z}$$

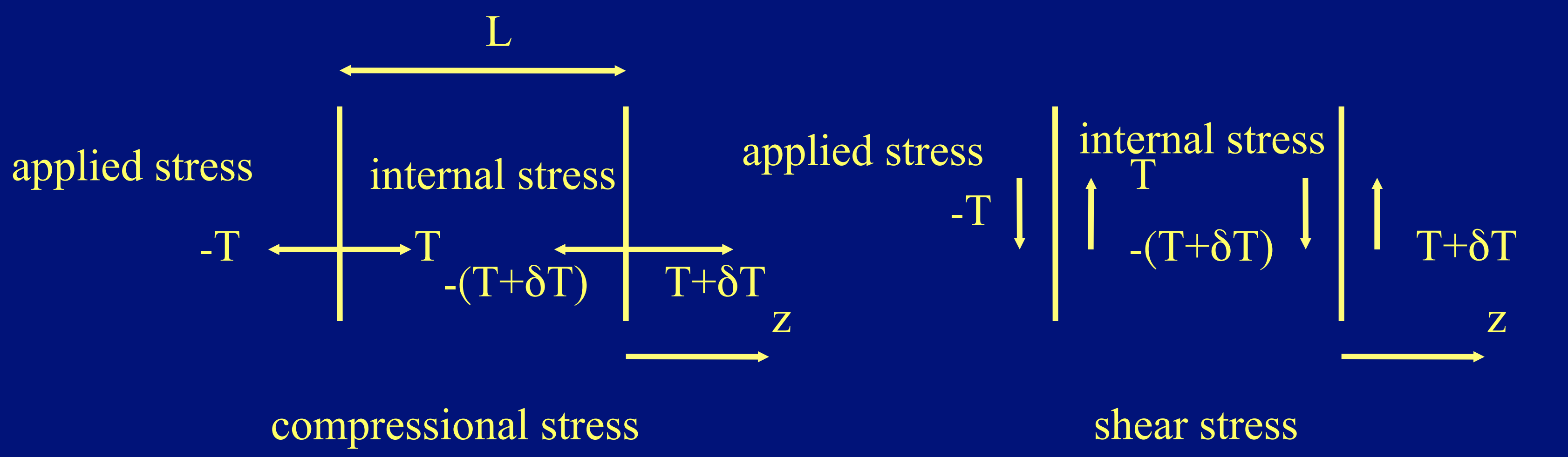
- Shear strain:

$$S \equiv \frac{\partial v}{\partial z}$$

# Stress (Pressure)

- Force per unit area applied to the object.
- Net force applied to a unit volume:

$$\partial T \,/\, \partial z$$



# Hooke's Law

- $T = cS$, where $c$ is the elastic constant.
- Tensor representation:

| Tensor notation | Reduced notation |
|---|---|
| *xx* | 1 |
| *yy* | 2 |
| *zz* | 3 |
| *yz*=*zy* | 4 |
| *zx*=*xz* | 5 |
| *xy*=*yx* | 6 |

# Hooke's Law (General Form)

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix}$$

- Stress tensor symmetry: no rotation.
- Strain tensor symmetry: by definition.

# Hooke's Law (Isotropy)

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix}$$

$$c_{11} = c_{12} + 2c_{44} = \lambda + 2\mu$$

- Lamé constants: λ and μ (shear modulus).

# Common Elastic Constants

- Young's modulus (elastic modulus, E):

$$T_{zz} = (\lambda + 2\mu) S_{zz} + \lambda (S_{xx} + S_{yy})$$
$$= \lambda (S_{xx} + S_{yy} + S_{zz}) + 2\mu S_{zz}$$
$$\equiv \lambda \Delta + 2\mu S_{zz}$$

( △: dilation )

$$E \equiv \frac{T_{zz}}{S_{zz}} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

$E \approx 3\mu$ for liquid and soft tissues

# Common Elastic Constants

- Bulk modulus (reciprocal of compressibility, B):

$$B \equiv -\frac{p}{\delta V/V} = -\frac{p}{\Delta}$$

$$p \equiv -\frac{(T_{xx} + T_{yy} + T_{zz})}{3} = -B \cdot \Delta$$

$$B = \frac{3\lambda + 2\mu}{3}$$

# Common Elastic Constants

- Poisson ratio (negative of the ratio of the transverse compression to the longitudinal compression, σ):

$$\sigma \equiv -\frac{S_{yy}}{S_{zz}} = \frac{\lambda}{2(\lambda + \mu)}$$

- σ approaches to 0.5 for liquid and soft tissues.

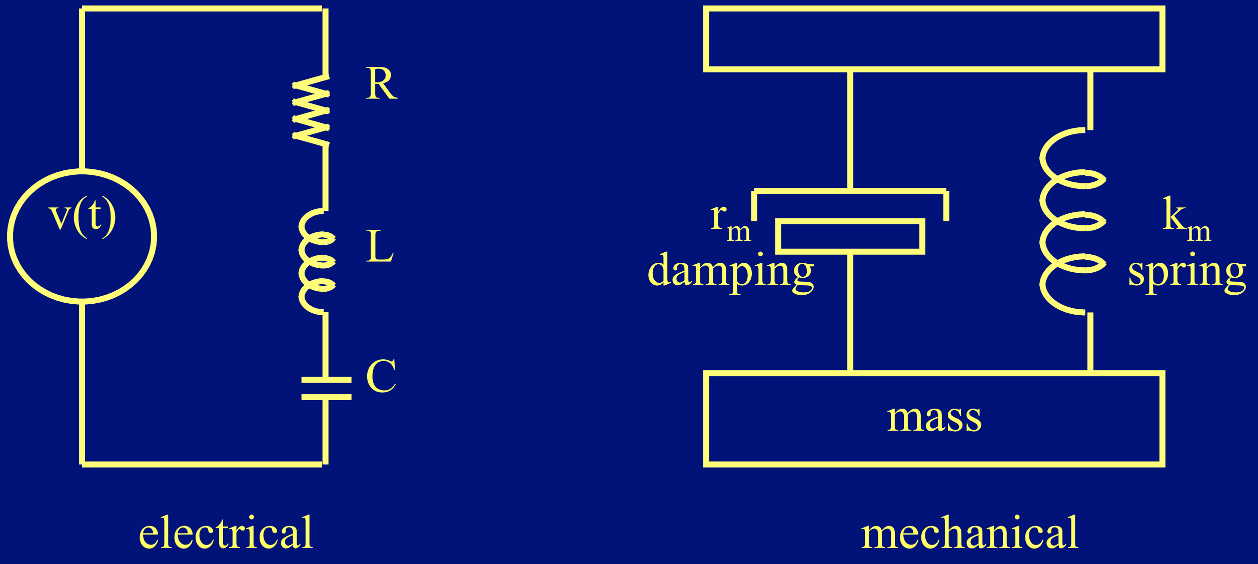
# Acoustic Wave Equation

# Acoustic Wave Equation

- Newton's second law
- Displacement → Particle velocity
- Particle velocity → Pressure
- Impedance = Pressure/Particle Velocity

# Acoustic Wave Equations

- Electrical and mechanical analogy:

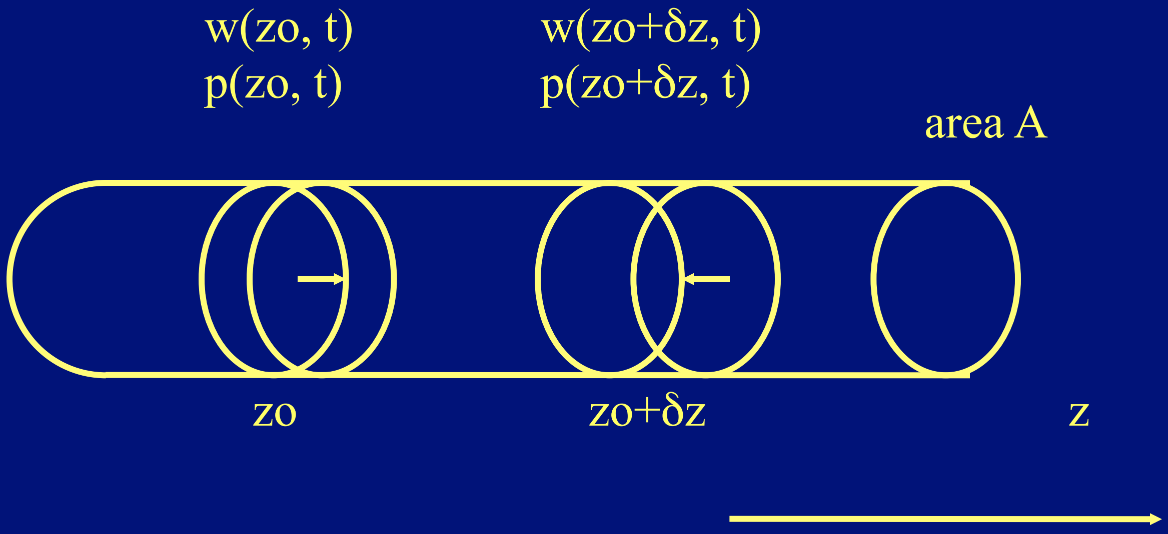


# Acoustic Wave Equations

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = v(t) \longleftrightarrow m\frac{d^2w}{dt^2} + r_m\frac{dw}{dt} + k_m w = f(t)$$

| Electrical | | Mechanical | |
|---|---|---|---|
| $q$ | charge | $w$ | displacement |
| $i=dq/dt$ | current | $U=dw/dt$ | particle velocity |
| $V$ | voltage | $f$ | force (stress, pressure) |
| $L$ | inductance | $m$ | mass |
| $1/C$ | 1/capacitance | $k_m$ | stiffness |
| $R$ | resistance | $r_m$ | damping |

# Acoustic Wave Equations




- Newton's second law:

$$A(p(z,t) - p(z+\delta z,t)) = (\rho \cdot \delta z \cdot A)\frac{\partial^2 w(z,t)}{\partial t^2}$$

$$\frac{\partial^2 w(z,t)}{\partial t^2} = (B/\rho)\frac{\partial^2 w(z,t)}{\partial z^2}$$

$$c = \sqrt{B/\rho}$$

# Acoustic Wave Equations

$$w(z,\omega) = w_1(\omega)e^{-j\omega z/c} + w_2(\omega)e^{j\omega z/c}$$

$$w(z,t) = w_1(t - z/c) + w_2(t + z/c)$$

$$u(z,t) \equiv \partial w(z,t)/\partial t$$

$$u(z,\omega) = j\omega w(z,\omega)$$

$$u(z,\omega) = u_1(\omega)e^{-j\omega z/c} + u_2(\omega)e^{j\omega z/c}$$

$$p(z,\omega) = -\frac{B}{j\omega}\frac{\partial u(z,\omega)}{\partial z}$$

$$= Z_0(u_1(\omega)e^{-j\omega z/c} - u_2(\omega)e^{j\omega z/c})$$

Characteristic impedance: $Z_0 = \rho c$

# Two Common Units

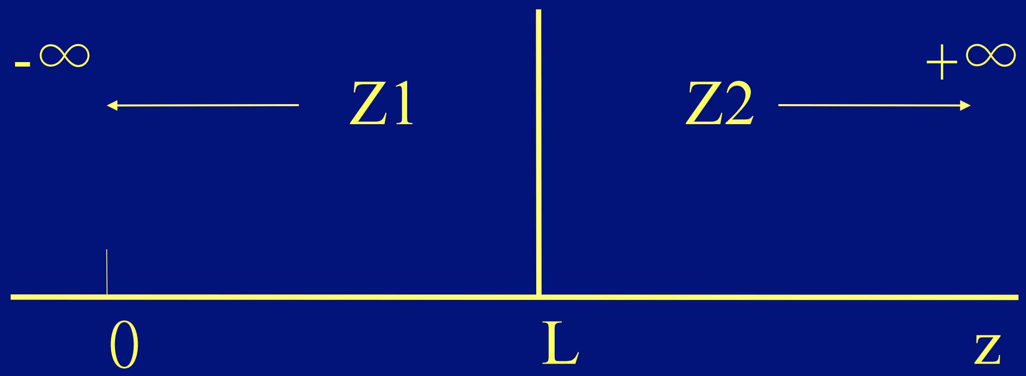
- Pa (Pascal, pressure) :

$$1Pa = 1N/m^2 = 1Kg/(m \cdot \sec^2)$$

- Rayl (acoustic impedance) :

$$1Rayl = 1Pa/(m/\sec) = 1Kg/(m^2 \cdot \sec)$$

# Reflection and Refraction




$$Z(z,\omega) \equiv \frac{p(z,\omega)}{u(z,\omega)} = Z_0 \frac{u_1(\omega)e^{-j\omega z/c} - u_2(\omega)e^{j\omega z/c}}{u_1(\omega)e^{-j\omega z/c} + u_2(\omega)e^{j\omega z/c}}$$

# Reflection and Transmission

- Pressure and particle velocity are continuous
- Medium 2 is infinite to the right

$$Z_1(L,\omega) = Z_2$$

$$Z_1 \frac{u_1(\omega)e^{-j\omega L/c} - u_2(\omega)e^{j\omega L/c}}{u_1(\omega)e^{-j\omega L/c} + u_2(\omega)e^{j\omega L/c}} = Z_2$$

# Reflection and Transmission

$$-u_2(\omega) = u_1(\omega)e^{-j\omega 2L/c}\frac{Z_2 - Z_1}{Z_2 + Z_1}$$

$$p(z,\omega) = Z_1 u_1(\omega)\left(e^{-j\omega z/c} + \frac{Z_2 - Z_1}{Z_2 + Z_1}e^{j\omega(z-2L)/c}\right)$$

At the boundary

$$p(L,\omega) = Z_1 u_1(\omega)e^{-j\omega L/c}\frac{2Z_2}{Z_2 + Z_1}$$

# Reflection and Transmission

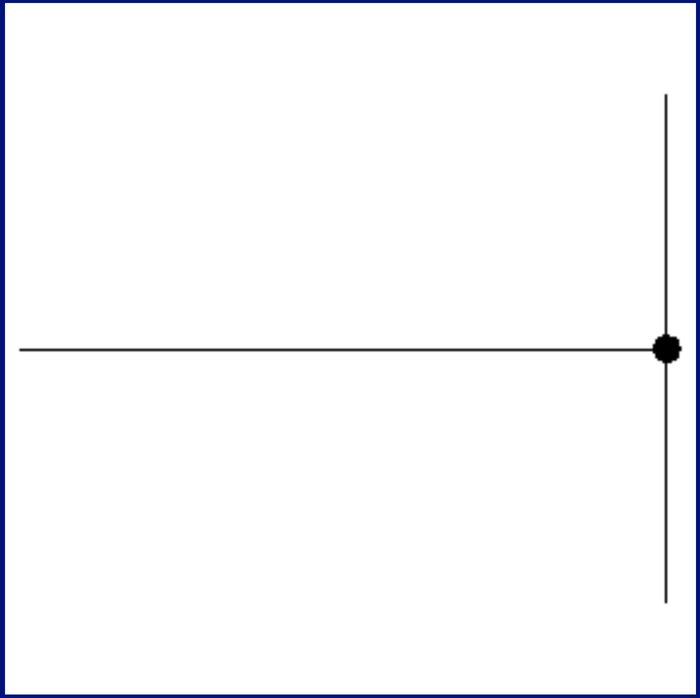
- 1D:

$$R_c = \frac{Z_2 - Z_1}{Z_2 + Z_1} \, (reflection)$$

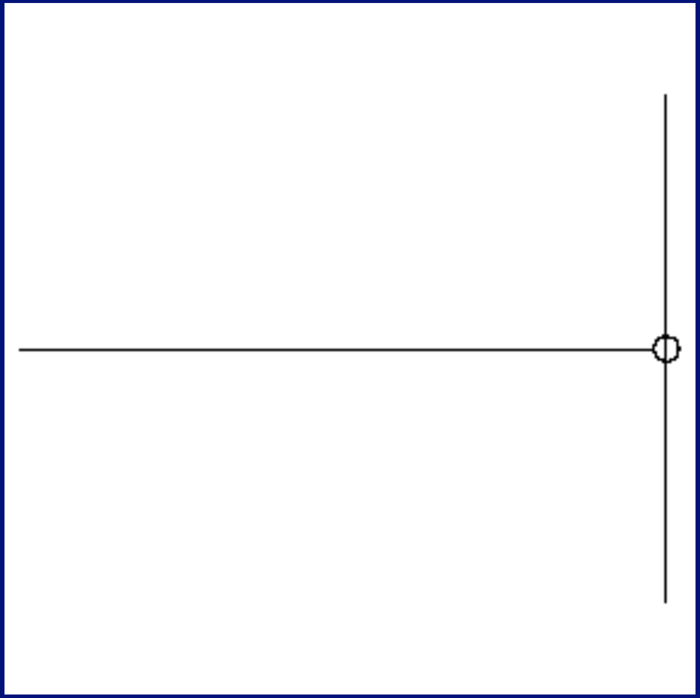
$$T_c = \frac{2Z_2}{Z_2 + Z_1} \, (transmission)$$

# Reflection

Hard Boundary

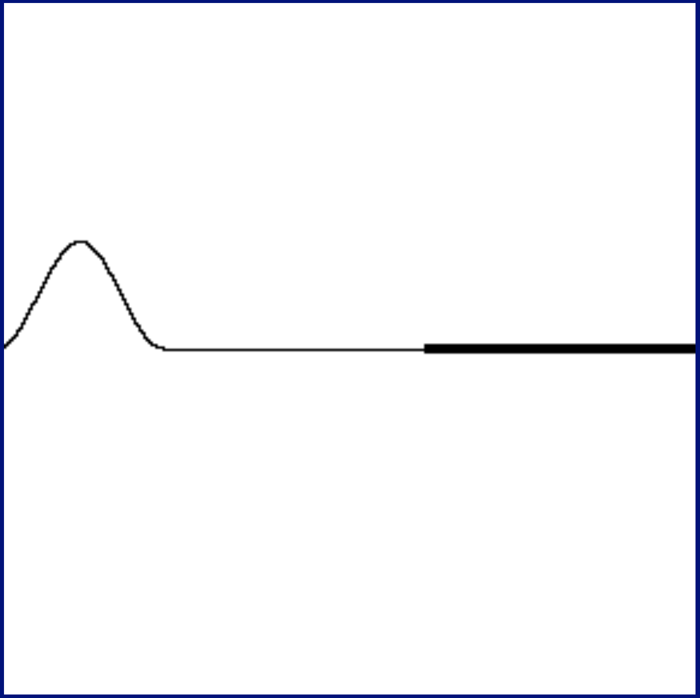


Soft Boundary

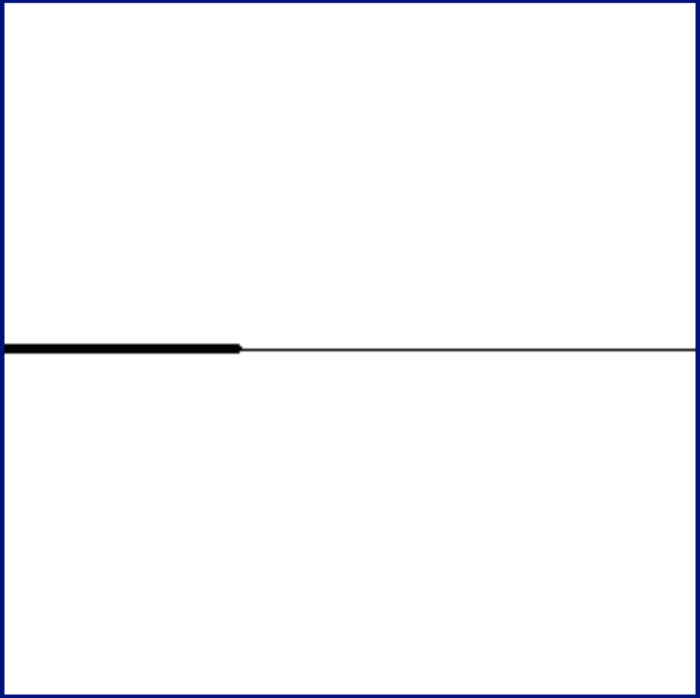


# Reflection

Low Density to High Density

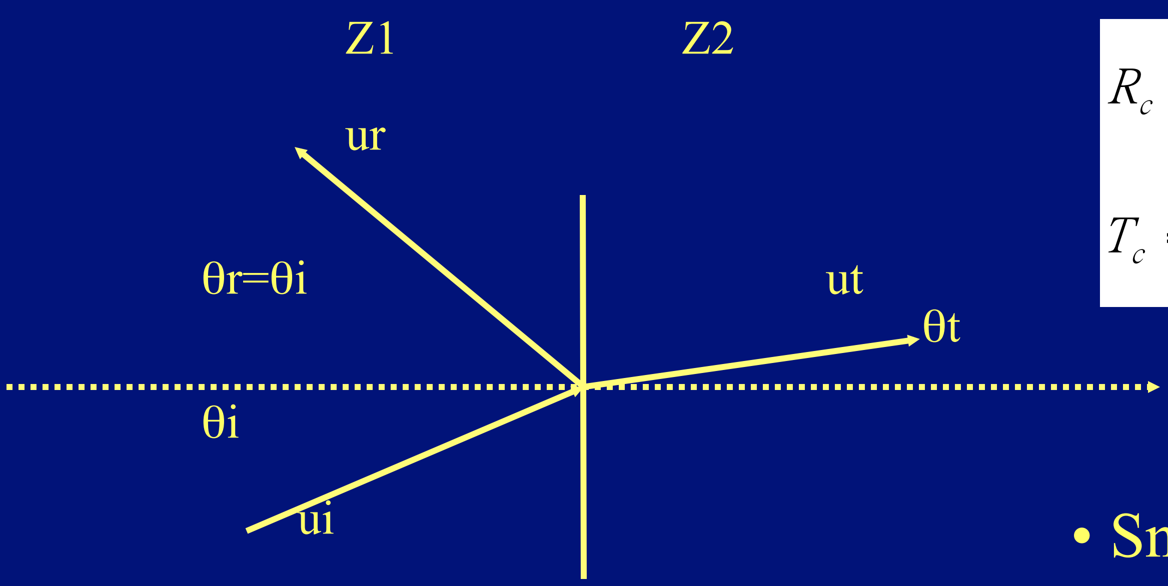


High Density to Low Density



# Reflection, Transmission and Refraction

- 2D:



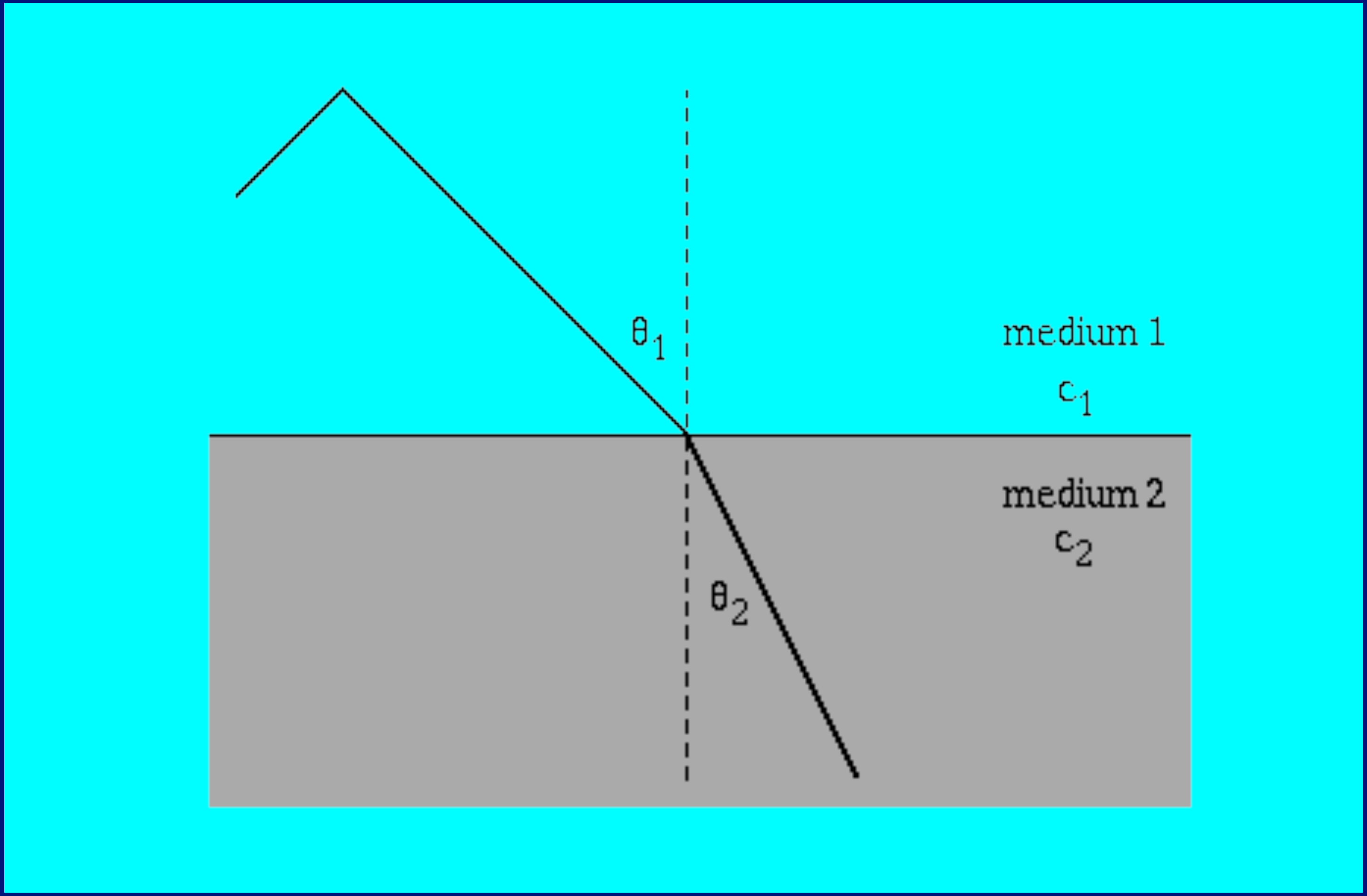

$$R_c = \frac{Z_2 \cos\theta_i - Z_1 \cos\theta_t}{Z_2 \cos\theta_i + Z_1 \cos\theta_t} (\mathit{reflection})$$

$$T_c = \frac{2Z_2 \cos\theta_i}{Z_2 \cos\theta_i + Z_1 \cos\theta_t} (\mathit{transmission})$$

- Snell's law: $\frac{\sin\theta_i}{\sin\theta_t} = \frac{c_1}{c_2}$

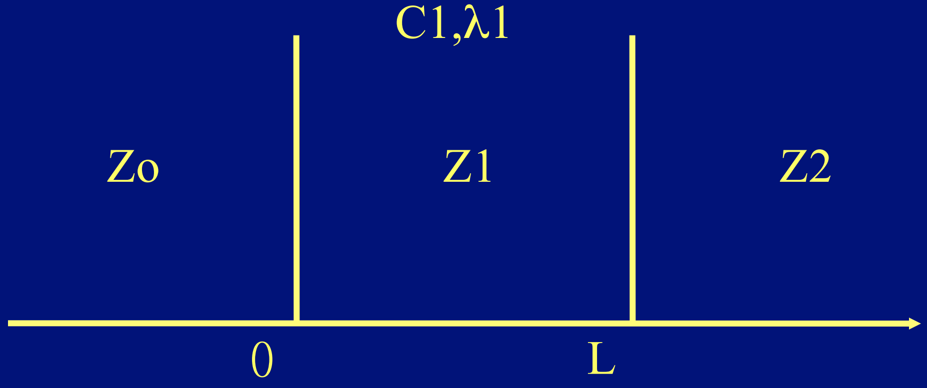
- Critical angle, if $c_2 \geq c_1$: $\theta_{ic} \equiv \sin^{-1}(c_1 / c_2)$

# Refraction



# Acoustic Impedance Matching

- Single layer




- In medium 1:

$$Z(z,\omega)=Z_1\frac{u_1(\omega)e^{-j\omega z/c}-u_2(\omega)e^{j\omega z/c}}{u_1(\omega)e^{-j\omega z/c}+u_2(\omega)e^{j\omega z/c}}$$

# Acoustic Impedance Matching

- Avoid reflection at the boundaries:

$$Z(0,\omega) = Z_0$$
$$Z(L,\omega) = Z_2$$

$\Rightarrow$

$$Z_0 = Z_1 \frac{Z_2 \cos\theta + jZ_1 \sin\theta}{Z_1 \cos\theta + jZ_2 \sin\theta}$$

$$\theta = \omega L / c_1 = 2\pi L / \lambda_1$$

For real $Z_o$:

$$L = (2n+1)\frac{\lambda_1}{4} \text{ for } n = 0,1,2\ldots$$

$$Z_1 = \sqrt{Z_0 Z_2}$$

- Double layer is common for medical transducers.

**Table IV**

Velocity and acoustic impedance of pertinent materials and biological tissues at room temperature (20–25°C)

| | Velocity (m/sec) | Impedance × $10^{-6}$ (kg/m$^2$-sec)[a] |
|---|---|---|
| Water | 1484 | 1.48 |
| Aluminum | 6420 | 17.00 |
| Air | 343 | 0.0004 |
| Plexiglas | 2670 | 3.20 |
| Blood | 1550 | 1.61 |
| Myocardium (perpendicular to fibers) | 1550 | 1.62 |
| Fat | 1450 | 1.38 |
| Liver | 1570 | 1.65 |
| Kidney | 1560 | 1.62 |
| Skull bone | 3360 (longitudinal) | 6.00 |

[a] Rayl is a unit commonly used for acoustic impedance. One rayl = 1 kg/m$^2$-sec.

TABLE IV. Acoustic and Piezoelectric Parameters

| Symbol | Definition |
|---|---|
| $d$ | Transmission constant – (strain out/field in) |
| $g$ | Receiving constant – (field out/stress in) |
| $\rho$ | Density |
| $v^e$ | Ultrasonic velocity in a particular direction $[(c^E/\rho)^{1/2}]$ |
| $Z_0$ | Characteristic acoustic impedance (lossless approximation) $(= \rho v)$ |
| $\epsilon^T$ | Free dielectric constant (unclamped) |
| $k_T$ | Electromechanical coupling efficiency $(k_T^2 = e^2/\epsilon^S c^E)$ |
| $Q_m$ | Mechanical quality factor |

TABLE V.[a] Material Properties

| | Longitudinal | | | | | |
|---|---|---|---|---|---|---|
| | Quartz (0° X-cut) | PZT-4[b] | PZT-5[b] | PZT-5H[b] | $PbNb_2O_6$[b] | $BaTiO_3$[b] |
| $d$ $(10^{-12}$ m/V) | 2 | 289 | 374 | 593 | 75 | 149 |
| $g$ $(10^{-3}$ Vm/N) | 50 | 26 | 25 | 20 | 35 | 14 |
| $\rho$ (kg/m$^3$) | 2650 | 7600 | 7500 | 7500 | 5900 | 5700 |
| $v^E$ (m/sec) | 5650 | 3950 | 3870 | 4000 | 2700 | 4390 |
| $Z_0$ $(10^6$ kg/m$^2$sec) | 15 | 30 | 29 | 30 | 16 | 25 |
| $\epsilon^T/\epsilon_0$ | 4.5 | 1300 | 1700 | 3400 | 240 | 1700 |
| $k_T$ (%) | 11 | 70 | 70 | 75 | 40 | 48 |
| $Q_m$ | >25000 | <500 | <75 | <65 | <5 | <400 |

**Table V**
Attenuation coefficients of biological tissues and pertinent materials

| Material | Attenuation coefficient (np/cm at 1 MHz at 20°C) |
|---|---|
| Air | 1.38 |
| Aluminum | 0.0021 |
| Plexiglas | 0.23 |
| Water | 0.00025 |
| Fat | 0.06 |
| Blood | 0.02 |
| Myocardium (perpendicular to fiber) | 0.35 |
| Liver | 0.11 |
| Kidney | 0.09 |
| Skull bone | 1.30 |

## *TABLE 9.3*

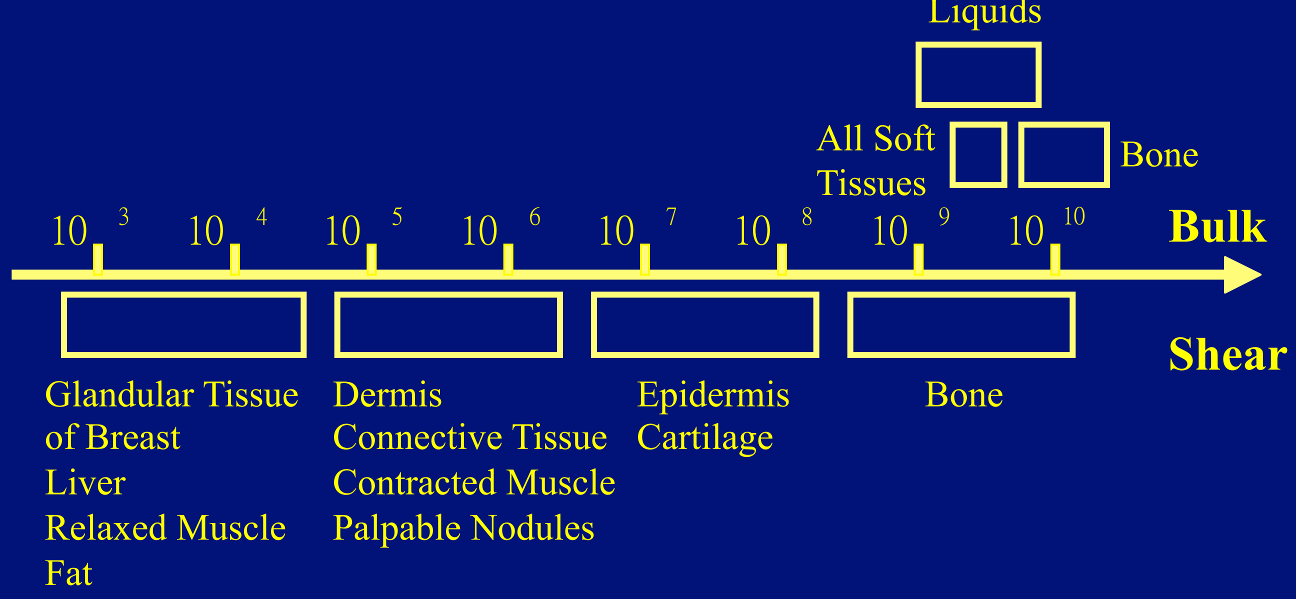
REFLECTIVITY OF NORMALLY INCIDENT WAVES

| Materials at Interface | Reflectivity |
|---|---|
| Brain–skull bone | 0.66 |
| Fat–bone | 0.69 |
| Fat–blood | 0.08 |
| Fat–kidney | 0.08 |
| Fat–muscle | 0.10 |
| Fat–liver | 0.09 |
| Lens–aqueous humor | 0.10 |
| Lens–vitreous humor | 0.09 |
| Muscle–blood | 0.03 |
| Muscle–kidney | 0.03 |
| Muscle–liver | 0.01 |
| Soft tissue (mean value)–water | 0.05 |
| Soft tissue–air | 0.9995 |
| Soft tissue–PZT5 crystal | 0.89 |

# Ultrasonic Elasticity Imaging

# Elasticity Imaging

- Image contrast is based on tissue elasticity (typically Young's modulus or shear modulus).



# Clinical Values

- Remote palpation.
- Quantitative measurement of tissue elastic properties.
- Differentiation of pathological processes.
- Sensitive monitoring of pathological states.

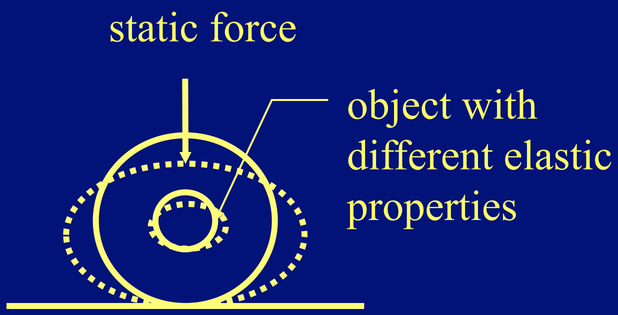
# Clinical Examples

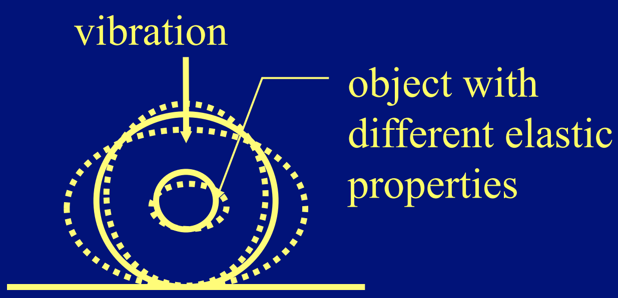
- Tumor detection by palpation: breast, liver and prostate (limited).
- Characterization of elastic vessels.
- Monitoring fetal lung development.
- Measurements of intraocular pressure.

# Basic Principles

- Hooke's law:
  - stress=elastic modulus*strain ($\sigma = C\varepsilon$).
- General approach:
  - Stress estimation.
  - Displacement measurement.
  - Strain Reconstruction.

# General Methods

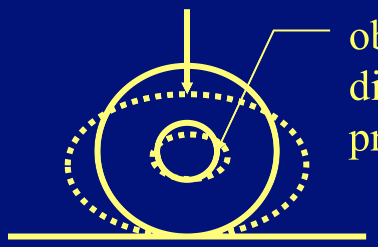



- Static or dynamic deformation due to externally applied forces.
- Measurement of internal motion.
- Estimation of tissue elasticity.

# Static Approaches

- Equation of equilibrium:




$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0, \;\; i = 1,2,3$$

where $\sigma_{ij}$ is the second order stress tensor, $f_i$ is the body force per unit volume in the $x_i$ direction.

# Dynamic Approaches

- Wave equation with harmonic excitation:

$$\mu \frac{\partial^2 U}{\partial x^2} = \rho \frac{\partial^2 U}{\partial t^2}$$

$$\mu = \rho c^2$$

- Doppler spectrum of a vibrating target:

$$s(t) = A\cos(\omega_x t)$$

$$\omega_x = \omega_0 + \Delta\omega_m \cos(\omega_L t)$$