Chapter 13 Laplace Transform Analysis

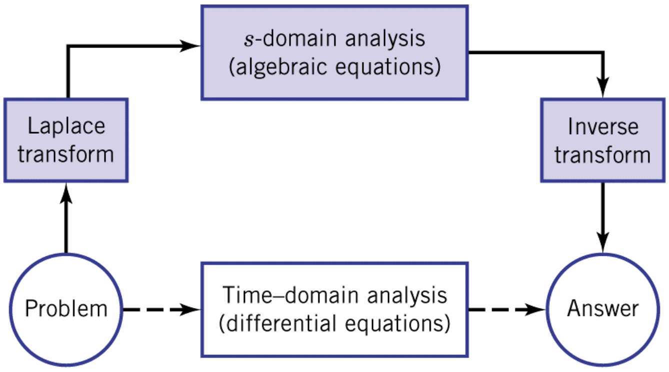


Figure 13.1

13.1 Laplace transforms

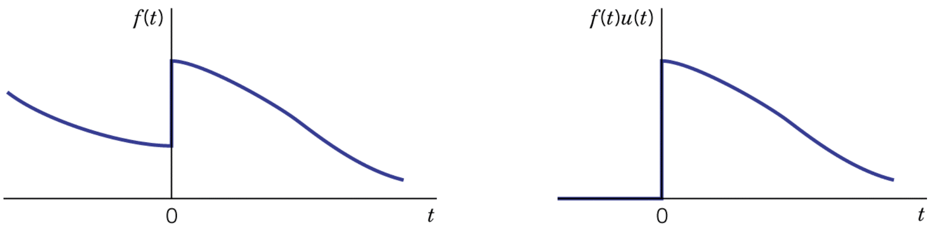
- *s*-domain phasor analysis: $x(t) = X_m e^{\sigma t} \cos(\omega t + \phi_x)$ ->

  $\underline{X} = X_m \angle \phi_x$.

- Laplace transform:

$$F(t) = L[f(t)] \equiv \int_{o^-}^{\infty} f(t) e^{-st} dt$$

- Three conditions:
  - ✓ Unilateral (one-sided Laplace transform): Laplace transform holds for $t \geq 0$. Previous effects are included in the initial conditions at $t = 0^-$.
  - ✓ Existence condition: $\lim_{t \to \infty} |f(t)| e^{-\sigma t} = 0$, $\sigma > \sigma_c$.
  - ✓ The resulting transform is a function of *s*.



(a) Waveform with discontinuity at $t = 0$ (b) One–sided waveform Figure 13.2.

- Inverse Laplace transform for $t \geq 0$:

$$f(t) = L^{-1}[F(s)] \equiv \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} F(s) e^{st} ds, \quad c > \sigma_c .$$

- Inverse Laplace transform is usually done by partial fraction expansion (will be covered in section 13.2).

Example 13.1

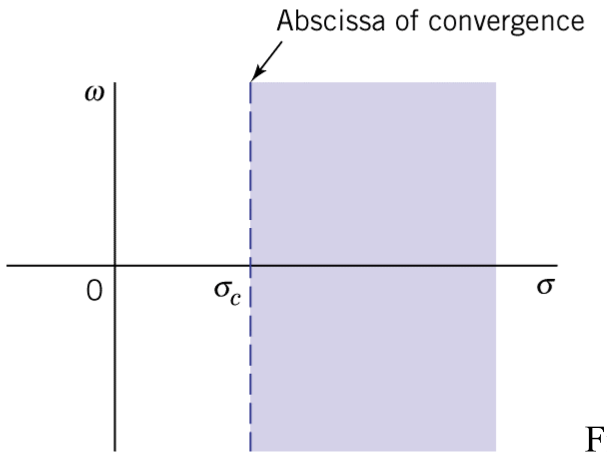


Figure 13.3.

Example 13.2

- Transform properties:
  - ✓ Linear combination:
  - ✓ Multiplication by e^{-st}:
  - ✓ Multiplication by t:
  - ✓ Time delay:

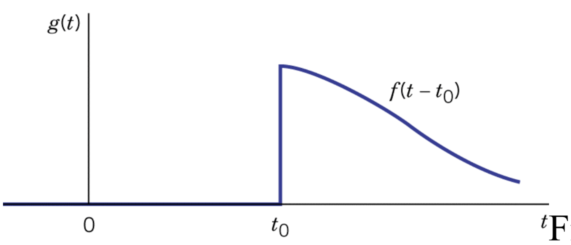
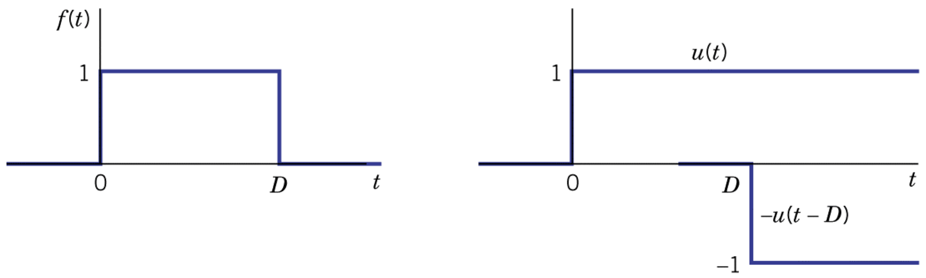


Figure 13.4.

✓ Differentiation and integration:

Example 13.3



(a) Rectangular pulse *(b)* Decomposition as two step functions Figure 13.5.

Example 13.4

(Please refer to Tables 13.1 and 13.2.)

- Solution of differential equations using Laplace transform:
    - ✓ The transformation automatically incorporates the initial conditions.
    - ✓ Transformation converts linear differential equations to s-domain algebraic equations.
    - ✓ Transformation is **similar** to the s-domain phasor analysis. Denominator of the s-domain function includes the characteristic polynomial.
    - ✓ Inverse transformation is required to obtain the resultant time domain function.

    - ✓ First-order example:

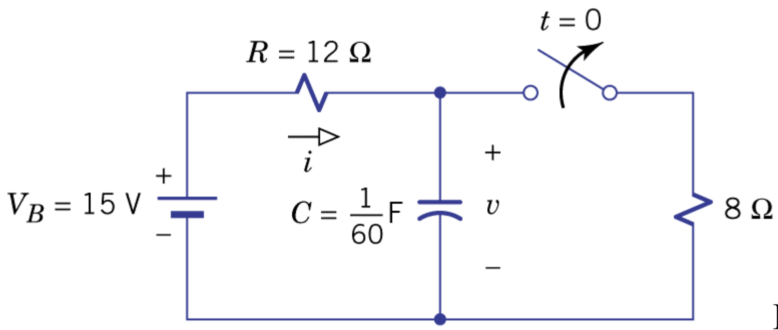


Figure 13.6.

✓ Second-order example:

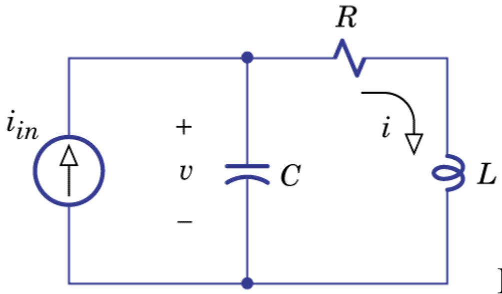


Figure 13.7.

13.2 Transform inversion

- Partial-fraction expansion of a strictly proper rational function

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}, \quad m \le n-1.$$

- Three cases will be considered: distinct real poles, complex poles and repeated poles.
- Case 1: distinct real poles

(Heaviside's theorem, cover-up rule.)

Example 13.5: Inversion of a Third-Order Function

$$I(s) = \frac{I_1 s^2 + (R/L) I_1 s + I_2 / LC}{s\left[s^2 + (R/L)s + 1/LC\right]}$$

- Case 2: complex poles

$$D(s) = (s^2 + 2\alpha s + \omega_0^2)(s - s_3)\cdots(s - s_n)$$

Example 13.6: Inversion with Complex Poles
(method of the undetermined coefficients)

$$F(s) = \frac{15s^2 - 16s - 7}{(s+2)(s^2 + 6s + 25)}$$

- Case 3: repeated poles

$$F(s) = G(s) + \frac{A_3}{s - s_3} + \cdots + \frac{A_n}{s - s_n}$$

$$G(s) = \frac{A_{i1}}{s - s_i} + \frac{A_{i2}}{(s - s_i)^2}$$

Example 13.7: Inversion with a Triple Pole

$$F(s) = \frac{-s^2 - 2s + 14}{(s+4)^3 (s+5)}$$

- Tim delay

Example 13.8: Inversion with Time Delay

- Initial-value theorem

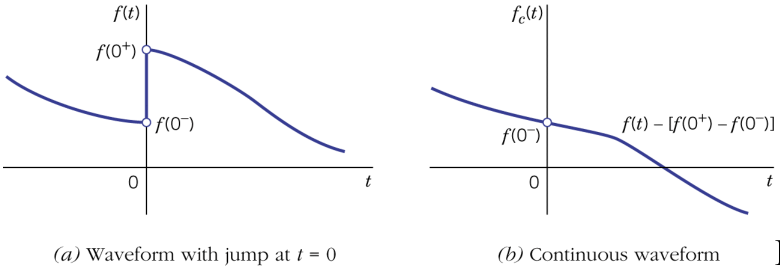


Figure 13.8.

- Final-value theorem

Example 13.9: Calculating Initial and Final Values

$$f(0^-) = 5\,,\quad F(s) = \frac{N(s)}{D(s)} = \frac{5s^3 - 1600}{s(s^3 + 18s^2 + 90s + 800)}$$

13.3 Transform circuit analysis

- Given a circuit with some initial state at $t = 0^-$ and an excitation $x(t)$ starting at $t = 0$, find the resulting behavior of any voltage or current $y(t)$ for $t \geq 0$.
- Zero-state response, natural response, forced response, zero-input response and complete response.
- Zero-state response: a circuit with no stored energy at $t = 0^-$.

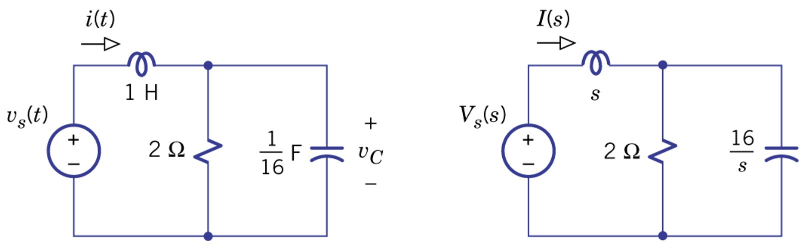
- Step response:

$$x(t) = u(t) = 1, \quad t > 0$$

$$X(s) = 1/s$$

$$Y(s) = H(s) \times \frac{1}{s} = \frac{N_H(s)}{sP(s)}$$

Example 13.10: Step Response



(a) Circuit for Example 13.10 *(b)* *s*-domain diagram Figure 13.9.

- Zero-state AC response:

$$x(t) = X_m \cos(\beta t + \phi_x)$$

$$D_X(s) = s^2 + \beta^2$$

Example 13.11: Zero-State AC Response

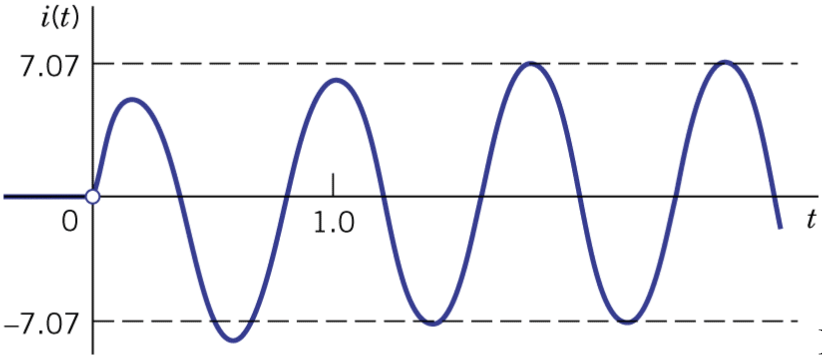
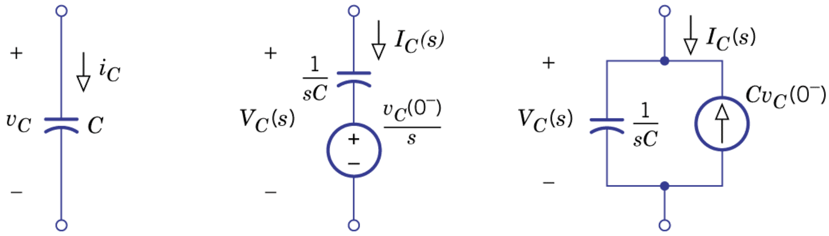


Figure 13.10.

- Natural response and forced response:

- Zero-input response: the excitation equals zero for $t \geq 0$ but the circuit contains stored energy at $t = 0^-$.

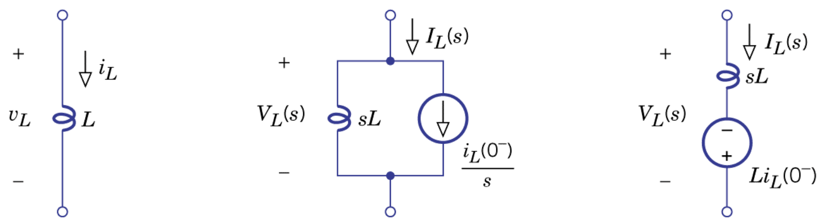


(a) Capacitor in the time domain

(b) *s*-domain Thévenin model of initial voltage

(c) *s*-domain Norton model of initial voltage

Figure 13.11.



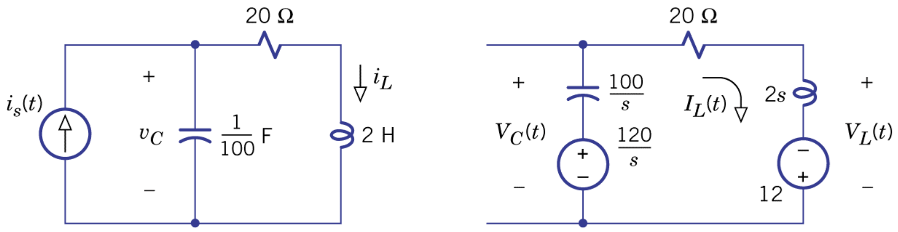
(a) Inductor in the time domain

(b) *s*-domain Norton model of initial current

(c) *s*-domain Thévenin model of initial current

Figure 13.12.

Example 13.12: Calculating a Zero-Input Response



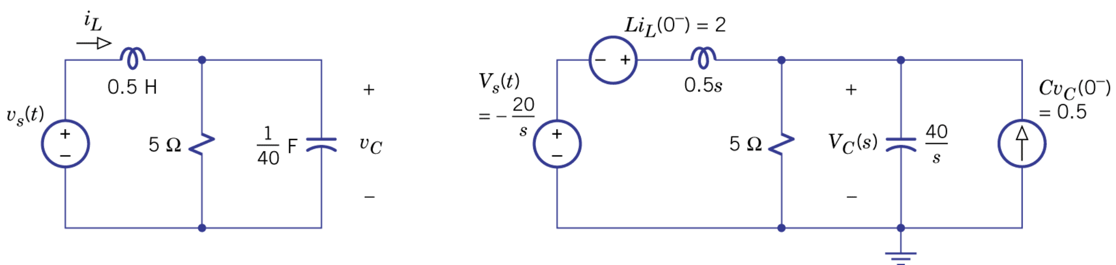
(a) Circuit for Example 13.12

(b) *s*-domain diagram with initial–value sources

Figure 13.13.

- Complete response: complete response=zero-input response + zero-state response.

Example 13.13: Calculating a Complete Response



(a) Circuit for Example 13.13

(b) *s*-domain diagram with input and initial–value sources

Figure 13.14.

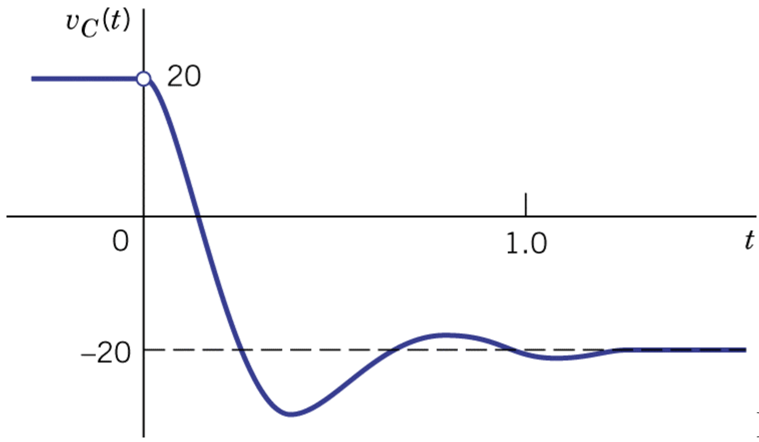


Figure 13.15.

13.4 Transform analysis with mutual inductance

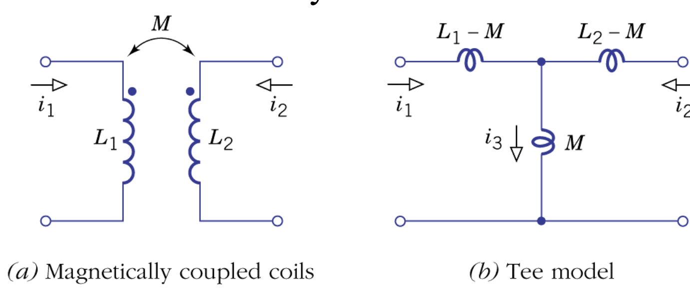


Figure 13.16.

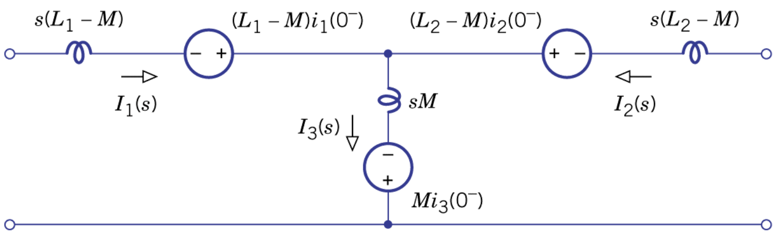
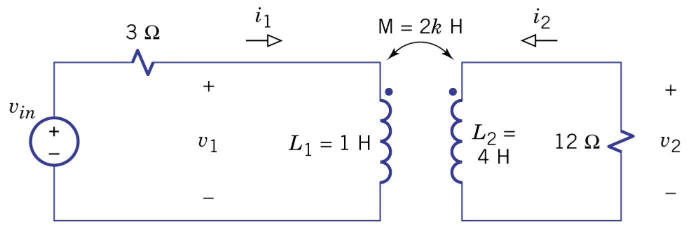
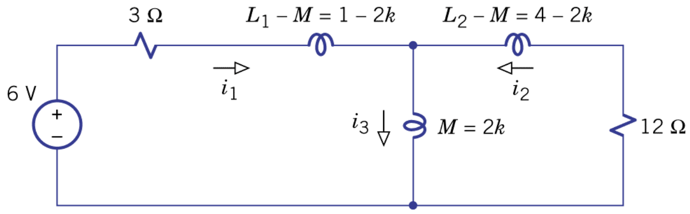


Figure 13.17.

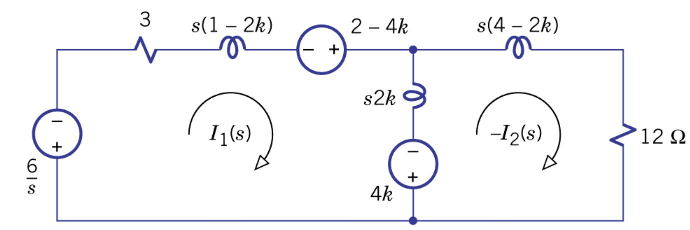
Example 13.14: Complete Response of a Transformer Circuit



(a) Transformer circuit



(b) Diagram with tee model for $t < 0$



(c) s–domain diagram for $t \geq 0$ with input and initial-value sources Figure 13.18.

13.5 Impulses and convolution

- Impulses:

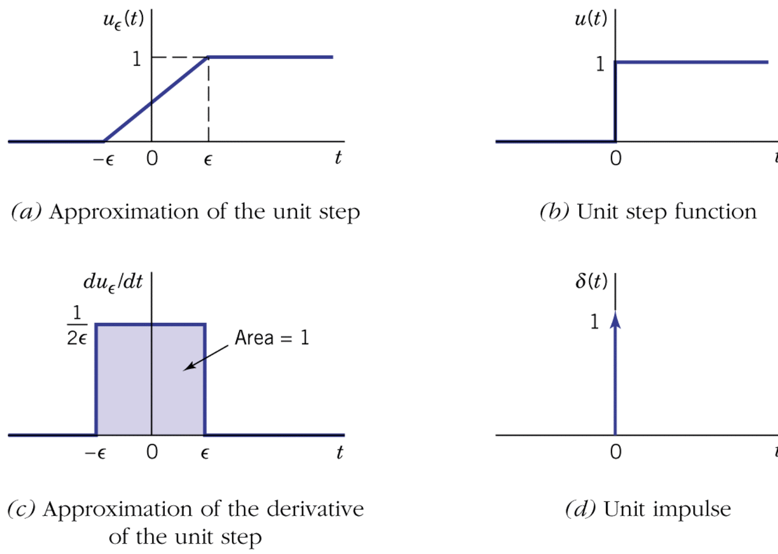


Figure 13.18.

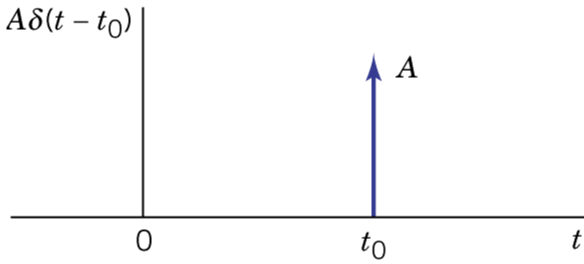
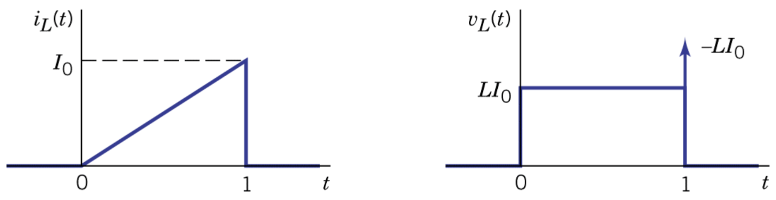


Figure 13.20.

- Sampling process:

Example 13.15: Discontinuous Inductor Current



(a) Inductor current waveform

(b) Voltage waveform with an impulse

Figure 13.21.

- Transform with impulses

$$L[\delta(t)] = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = e^{-st}\Big|_{t=0} = 1.$$

- As an example, the inverse transform of $\dfrac{3s}{s+2}$ is

- In general, the inverse of $F(s)$

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

Example 13.16: Impulsive Zero-State Response

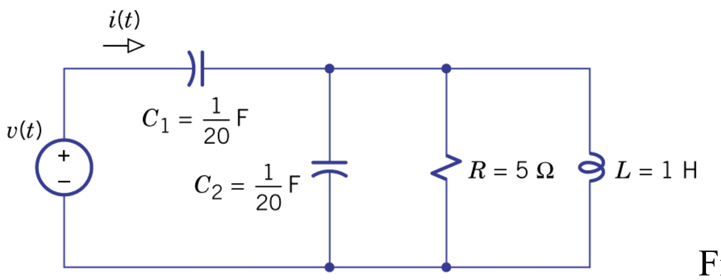


Figure 13.22.

- Convolution and impulse response.
- The convolution of $f(t)$ and $g(t)$:

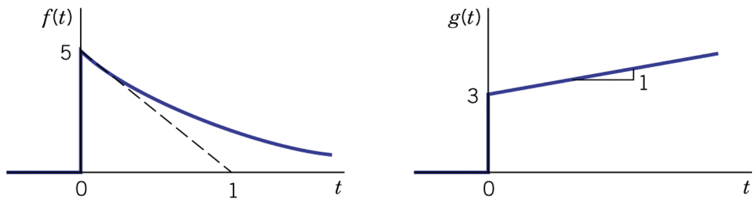
$$f(t) * g(t) \equiv \int_{-\infty}^{\infty} f(\lambda) g(t-\lambda) d\lambda .$$

- Commutative property ($f(t) * g(t) \equiv g(t) * f(t)$):

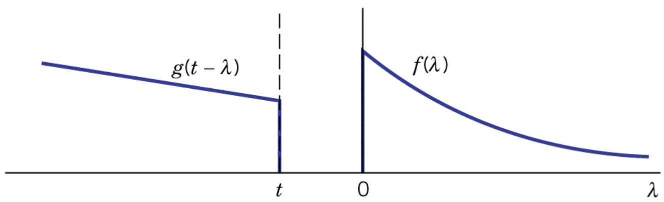
- Distributive property ($f(t) * [g_1(t) + g_2(t)] \equiv [f(t) * g_1(t)] + [f(t) * g_2(t)]$):

- Replication property ($f(t) * \delta(t) \equiv f(t)$):

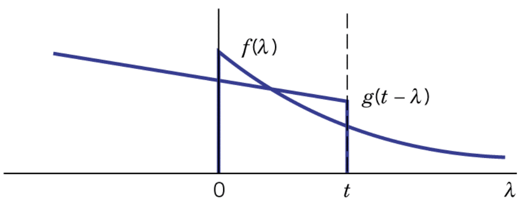
- Graphical interpretation:



(a) Waveforms of $f(t)$ and $g(t)$



(b) $f(\lambda)$ and $g(t-\lambda)$ with $t < 0$



(c) $f(\lambda)$ and $g(t-\lambda)$ with $t > 0$

Figure 13.23.

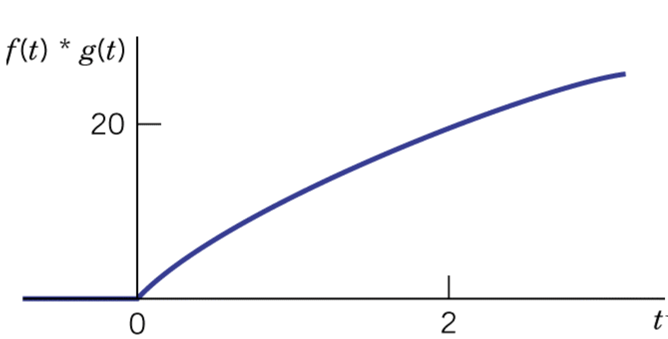


Figure 13.24.

- Convolution of two causal functions produces another causal function.

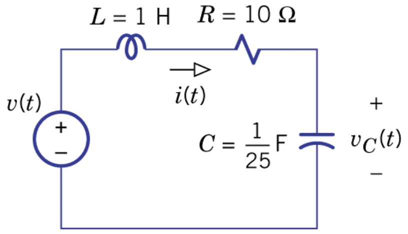
- Laplace transform of the convolution of causal functions:

$$L[f(t)*g(t)] = F(s)G(s)$$

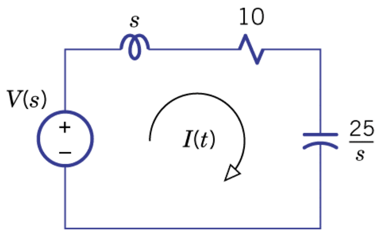
$$L^{-1}[F(s)G(s)] = f(t)*g(t)$$

(a) s-domain block diagram (b) Time-domain block diagram Figure 13.25.

Example 13.17: Impulse Response and Pulse Response

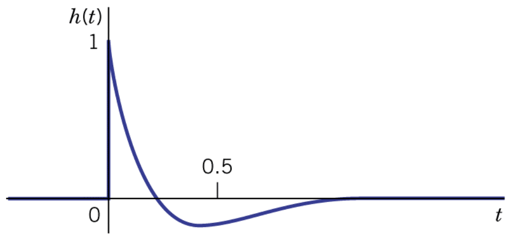


(a) Circuit for Example 13.17

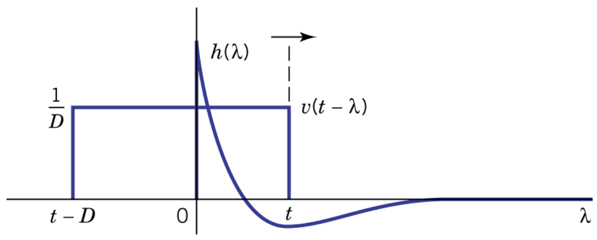


(b) s-domain diagram

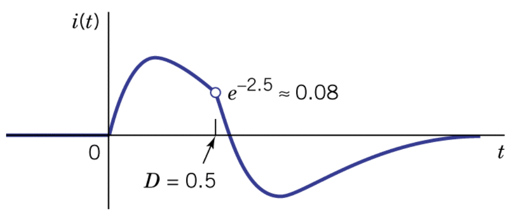
Figure 13.26.



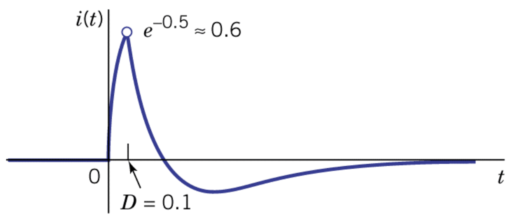
(a) Impulse response



(b) Convolution with rectangular pulse



(c) Pulse response with $D = 0.5$



(d) Pulse response with $D = 0.1$

Figure 13.27.