# Chapter 7 AC Power and Three-Phase Circuits

# Chapter 7: Outline

Power and Energy: **R: Dissipated; L, C: Stored.**

↓

R (real) → Z (complex); Real Power → Complex Power

$$Z = R + jX \rightarrow S = P + jQ$$

Resistance Reactance ↓ Real power Reactive power

DC vs. AC → Peak value vs. RMS value

↓

Power Transfer: Impedance Matching

$(R_L = R_S \rightarrow Z_L = Z_S^*)$

↓

Power Transfer Efficiency → Power Factor Correction

# Power in AC Circuits

# Power and Energy

- Given instantaneous power $p(t)$, the total energy $w$ transferred to a load between $t_1$ and $t_2$ is:

$$w = \int_{t_1}^{t_2} p(t)\,dt$$

- The average power $P$ is the average rate of energy transfer defined as:

$$P \equiv \frac{w}{t_2 - t_1} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} p(t)\,dt.$$

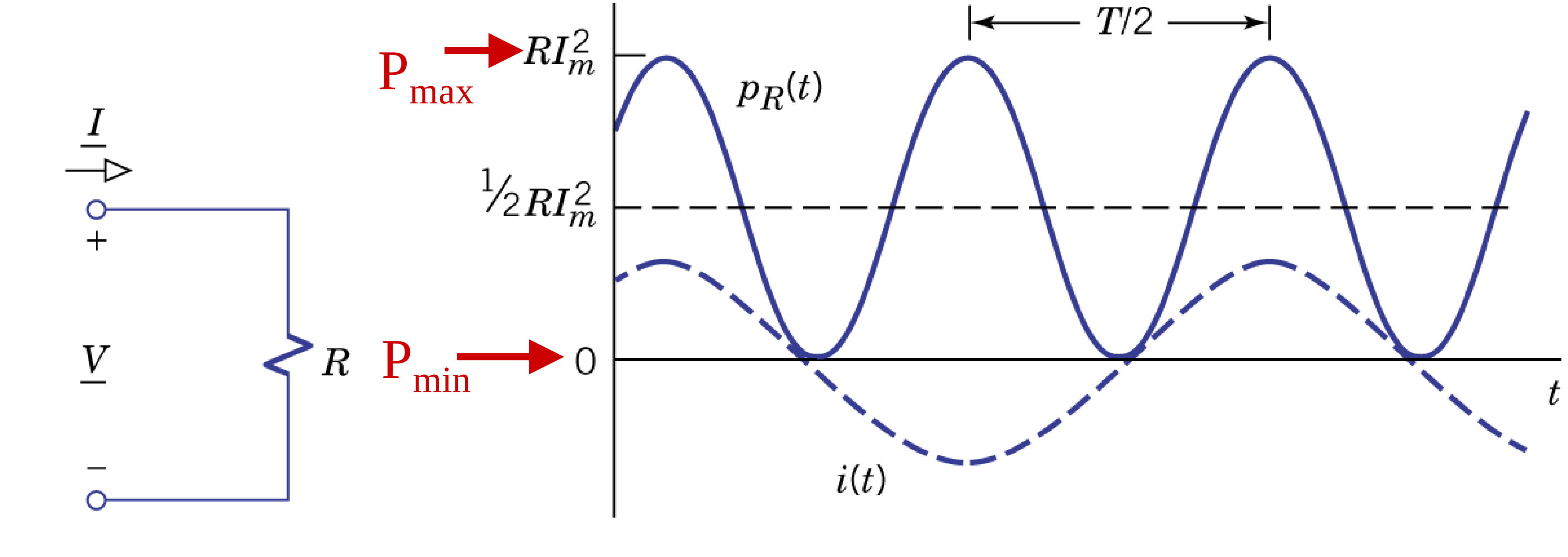
# Average Power

- Let *T* stand for any integer multiple of the period of *p*(*t*), the average power over *T* is the same as the long term average by letting t2→∞.
- The long term average is

$$P \equiv \frac{w}{T} = \frac{1}{T}\int_{t_1}^{t_1+T} p(t)dt$$

# Average Power of a Periodic Function

- Suppose $p(t)$ consists of a constant component and a periodic function, its long term average is equal to the constant component.




(a) Resistor in the ac steady state

(b) Waveforms of $i(t)$ and $p_R(t) = Ri^2(t)$

# Average Dissipated Power

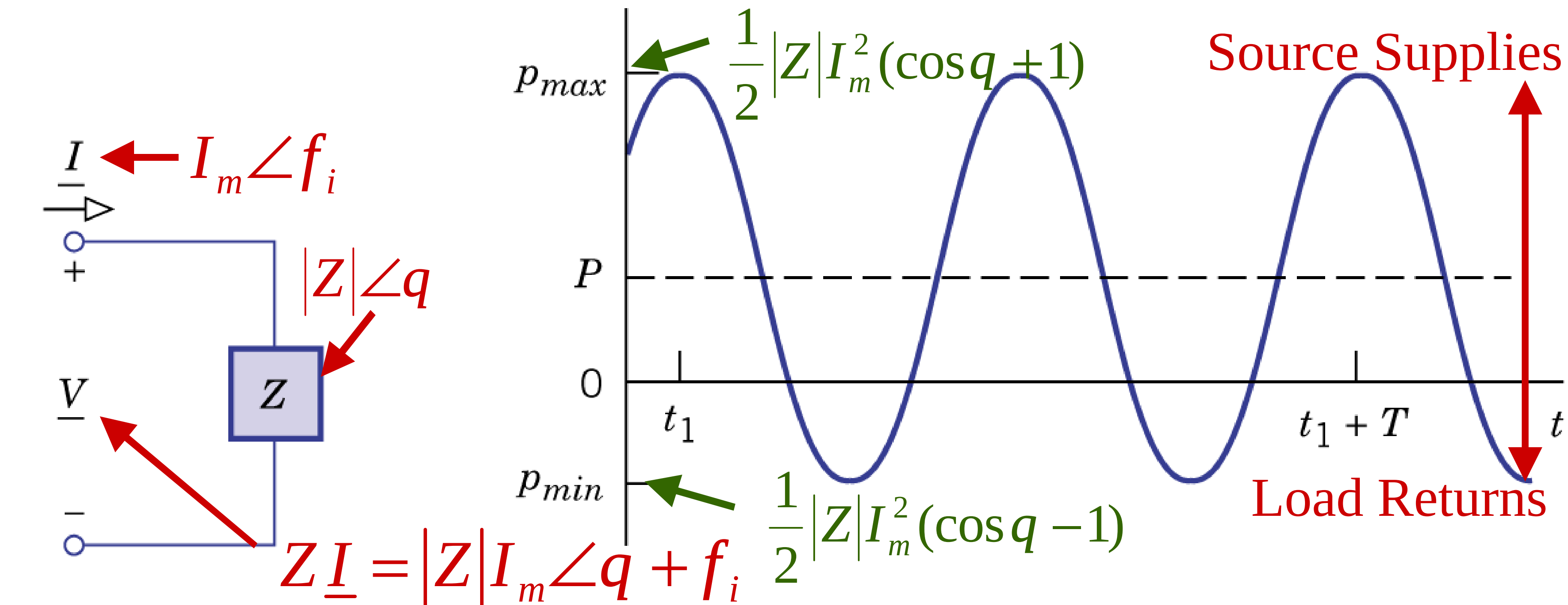
- The average power $P_R$ dissipated by a resistor $R$ with a peak current $I_m$ and a peak voltage $V_m$ is:

$$P_R(t) = Ri^2(t) = RI_m^2 \cos^2(wt + f_i) = \frac{1}{2} RI_m^2 \left(1 + \cos(2wt + 2f_i)\right)$$

$$\underline{I} = \frac{\underline{V}}{R}, I_m = \frac{V_m}{R} \qquad P_R = 1/2 RI_m^2 = \frac{V_m^2}{2R}$$

# Average Dissipated Power

- With an arbitrary load:




(a) Arbitrary load network

(b) Waveform of $p(t)$

# Average Dissipated Power

$$v(t) = |Z| I_m \cos(wt + q + f_i)$$

$$i(t) = I_m \cos(wt + f_i)$$

$$p(t) = v(t)i(t) = v(t) = |Z| I_m^2 \cos(wt + q + f_i)\cos(wt + f_i)$$

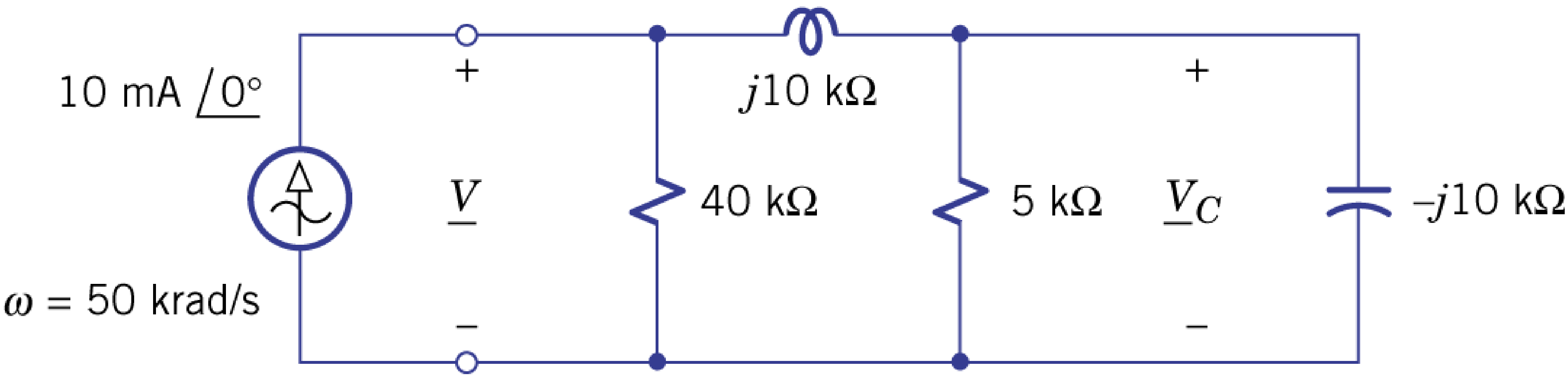
$$= \frac{1}{2}|Z| I_m^2 \left[\cos q + \cos(2wt + q + 2f_i)\right]$$

- The average power dissipated by the load is:

$$P = 1/2|Z| I_m^2 \cos q = 1/2 R(w) I_m^2 = \frac{R(w)}{2|Z|^2} V_m^2$$

$$R(w) = |Z|\cos q, \quad I_m = \frac{V_m}{|Z|}$$

# Example 7.1 AC Power Calculations




$$Z = 4.8k + j6.4k$$

$$|\underline{V}| = 80V$$

$$|\underline{V}_C| = 40V$$

$$P = \frac{1}{2} R(w) I_m^2 = \frac{1}{2}(4.8k)(10m)^2 = 240mW$$

$$P_{40} = \frac{80^2}{2 \cdot 40} = 80mW, \; P_5 = \frac{40^2}{2 \cdot 5} = 160mW$$

$$P = P_{40} + P_5$$

For a constant current through a resistance:

$$P = \frac{1}{T}\int_{t_1}^{t_1+T} p(t)dt = R\left[\frac{1}{T}\int_{t_1}^{t_1+T} i^2(t)dt\right] \equiv RI^2$$

# Root Mean Square Value

- An effective constant current $I$ to $i(t)$ with respect to power dissipation has the following form:

$$I^2 = \frac{1}{T}\int_{t_1}^{t_1+T} i^2(t)dt$$

- For a periodic current, the effective current is equal to the root mean square (rms) value. The root-mean-square is defined as

$$I_{rms} \equiv \sqrt{\frac{1}{T}\int_{t_1}^{t_1+T} i^2(t)dt} = \frac{I_m}{\sqrt{2}} \longleftarrow \text{Sinusoidal}$$

- Likewise, the rms value of a periodic voltage is

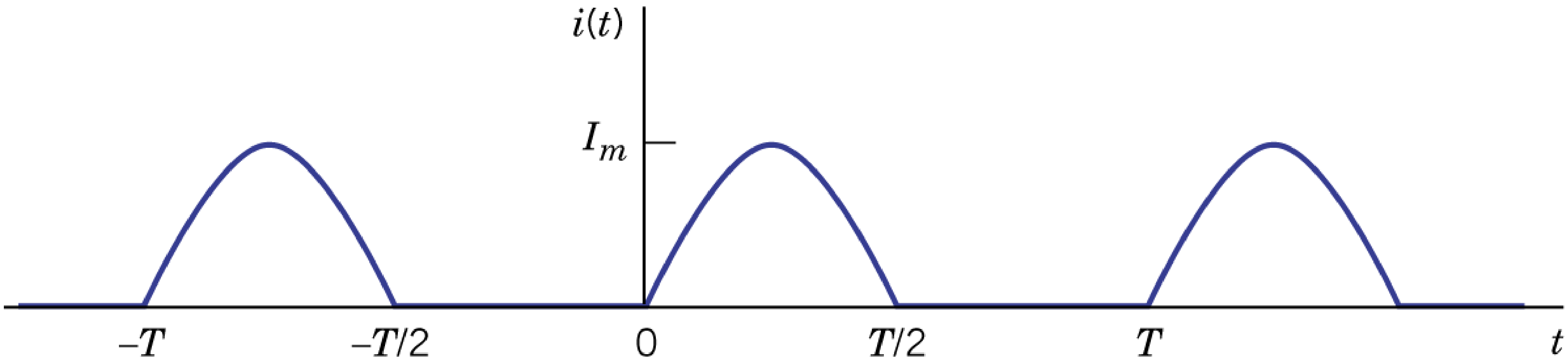
$$V_{rms} \equiv \sqrt{\frac{1}{T}\int_{t_1}^{t_1+T} v^2(t)dt} = \frac{V_m}{\sqrt{2}} \longleftarrow \text{Sinusoidal}$$

# Root Mean Square Value

$$P = R(w) I_{rms}^2 = \frac{R(w)}{|Z|^2} V_{rms}^2 .$$

$$V_{rms} = |Z| I_{rms}$$

# Example 7.2: RMS Value of a Half-Rectified Wave



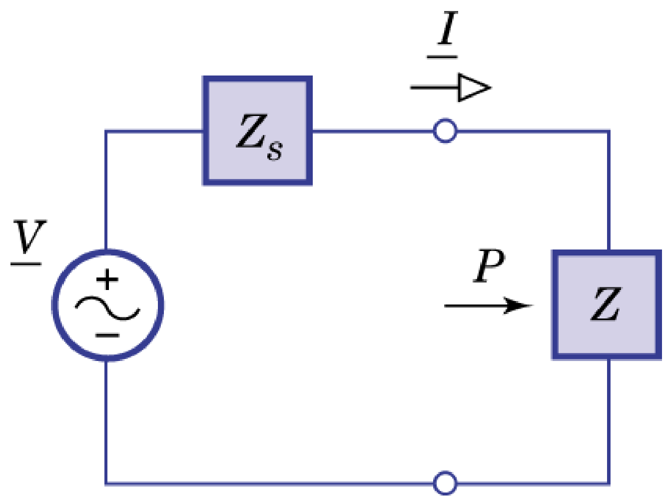

$$i(t) = I_m \sin wt \quad \text{for } 0 \le t \le \frac{T}{2}$$

$$= 0 \quad \text{for } \frac{T}{2} < t < T$$

$$I_{rms}^2 = \frac{1}{T}\int_0^{T/2} I_m \sin^2 \frac{2pt}{T} dt = \frac{I_m^2}{4}$$

$$I_{rms} = I_m / 2, \quad P_R = RI_{rms}^2 = RI_m^2 / 4$$

# Maximum Power Transfer (vs. Maximum Efficiency)




$$Z_s = R_s + jX_s,\; Z = R + jX$$

$$P = RI_{rms}^2,\quad P_s = R_s I_{rms}^2$$

$$Eff \equiv \frac{P}{P_s + P} = \frac{R}{R_s + R}$$

$$P = \frac{RV_{rms}^2}{\left|Z_s + Z\right|^2} = \frac{RV_{rms}^2}{(R_S + R)^2 + (X_s + X)^2}$$

For maximum power transfer and $R > 0$

$$\Rightarrow X = -X_s,\; R = R_s$$

# Maximum Power Transfer

- When

$$Z = R_s - jX_s = Z_s^*$$

the impedances are matched for maximum power transfer.

- Maximum available power (50% efficiency):

$$P_{\max} = V_{rms}^2 / 4R_s$$

# Maximum Power Transfer

- If the ratio $X/R$ is fixed but $|Z|$ can be adjusted, the maximum value of $|Z|$ is $|Z|=|Z_s|$.

$$R = |Z|\cos q, \quad X = |Z|\sin q$$

$$P = \frac{|Z|\cos q V_{rms}^2}{|Z_s|^2 + 2|Z|(R_s \cos q + X_s \sin q) + |Z|^2}$$

$$\text{Let } \frac{dP}{d|Z|} = 0, \text{ we have } |Z| = |Z_s|$$

# Example 7.3: Power Transfer from an Oscillator

$$V_{rms} = 1.2V, \quad Z_s = 6 + j8 = 10k\Omega\angle 53.1^0$$

$$\text{Case 1: Matched load impedance}$$

$$Z = Z_s^* = 6 - j8k\Omega$$

$$P_{\max} = \frac{1.2^2}{4 \cdot 6} = 60mW, \; Eff = 50\%$$

$$\text{Case 2: } \frac{\text{X}}{\text{R}} = -\frac{7}{24}$$

$$Z = (24 - j7)c = 25c\angle 16.3^0$$

$$|Z| = |Z_s|, \quad Z = 10k\Omega\angle 16.3^0$$

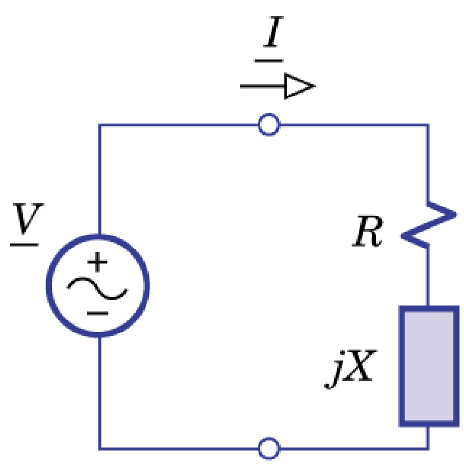
$$P = 51.1mW, \; Eff = 62.2\%$$

# Power Systems

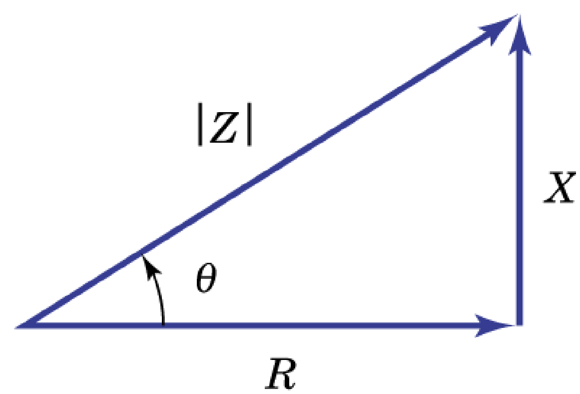
# Load in a AC Power System

- For ac power systems operating at a fixed frequency (e.g., 2p×60Hz), frequency-dependent effects can be ignored. In this case, any load impedance can be written as:

$$Z = |Z|\angle q = R + jX = |Z|\cos q + j|Z|\sin q$$



*(a)* Series load model



*(b)* Impedance triangle

# Phasors in RMS

$$\underline{V} = V_{rms} \angle \phi_v$$

$$\underline{I} = I_{rms} \angle \phi_i = I_{rms} \angle \phi_v - \theta$$

$$\underline{V} = Z \, \underline{I}$$

For Sinusoids:

$$V_m = \sqrt{2} V_{rms} = \sqrt{2} |\underline{V}|$$

$$I_m = \sqrt{2} I_{rms} = \sqrt{2} |\underline{I}|$$

# Real Power and Reactive Power

- Instantaneous power $(\frac{1}{2}|Z|I_m^2 = V_{rms} I_{rms})$ :

$$p(t) = \frac{1}{2}|Z|I_m^2\left[\cos q + \cos(2wt + 2f_v - q)\right]$$

$$= p_R(t) + p_X(t) = P[1 + \cos 2(wt + f_v)] + Q\sin 2(wt + f_v)$$

- Real power (average absorbed power in W):

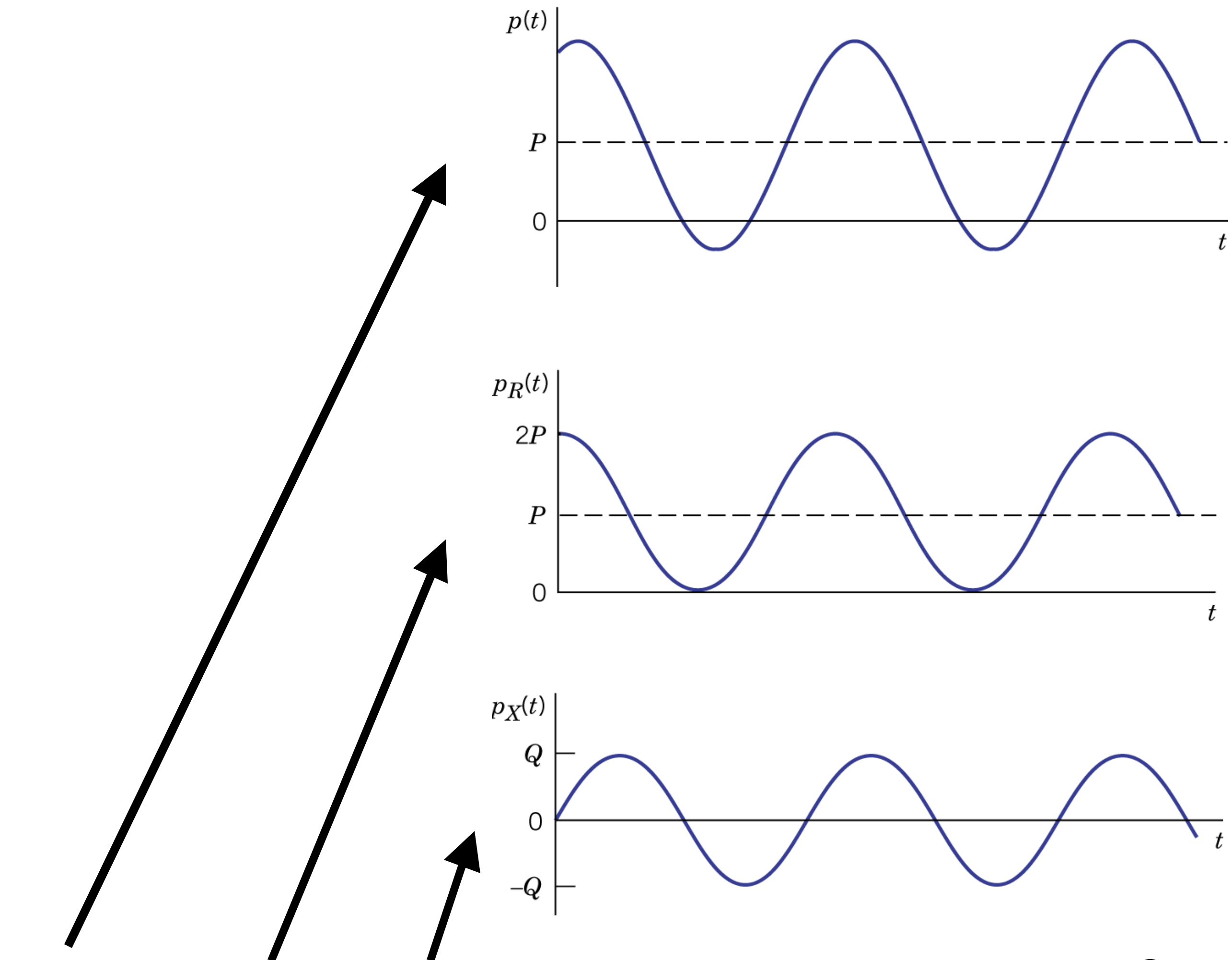
$$P = V_{rms} I_{rms} \cos\theta = R I_{rms}^2$$

- Reactive power (rate of energy exchange in VAr):

$$Q = V_{rms} I_{rms} \sin\theta = X I_{rms}^2$$

Voltage-amperes reactive

# Real Power and Reactive Power




$$p(t) = p_R(t) + p_X(t) = P[1 + \cos 2(wt + f_v)] + Q \sin 2(wt + f_v)$$

# Inductive vs. Capacitive

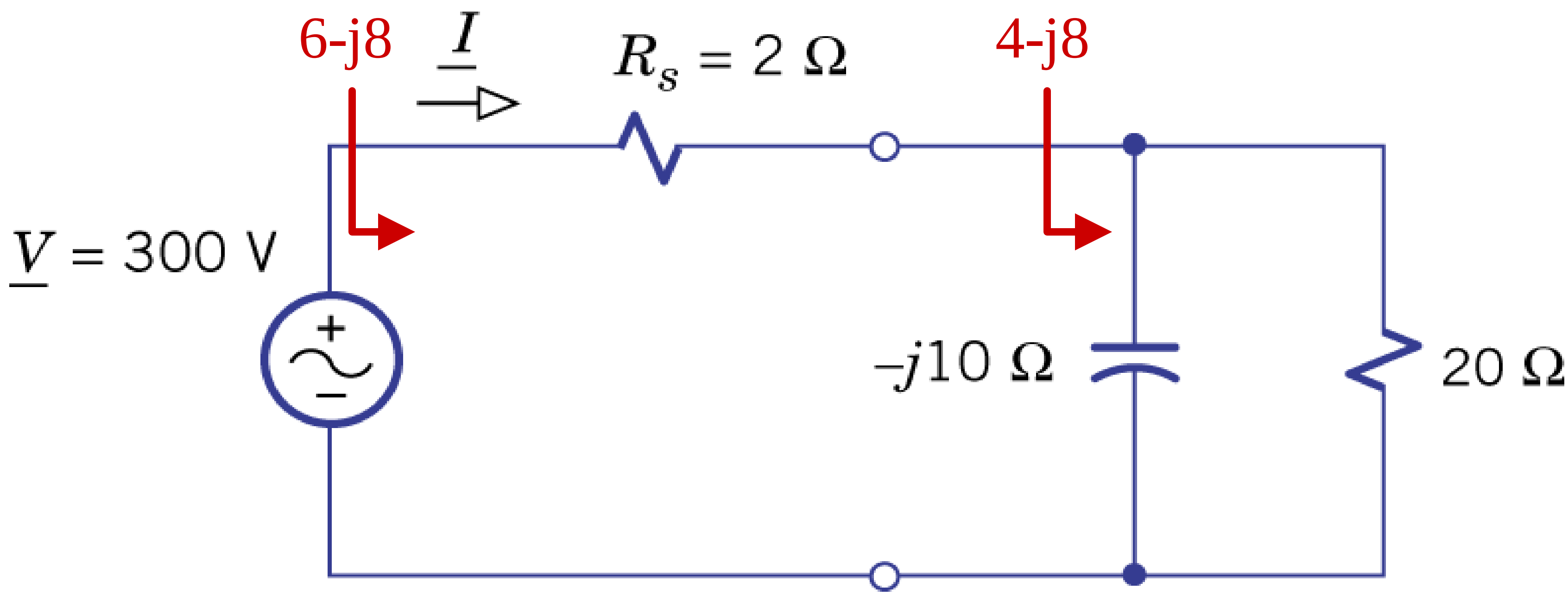
For a single inductor: $Q_L = wL\left|\underline{I}_L\right|^2 = \dfrac{\left|\underline{V}_L\right|^2}{wL}$

For a single capacitor: $Q_C = -\dfrac{\left|\underline{I}_C\right|^2}{wC} = -wC\left|\underline{V}_C\right|^2$

- A load with inductive reactance (*X>0*), *Q>0*
- A load with capacitive reactance (*X<0*), *Q<0*

<u>Reactive power increases the rms current to achieve the same average power → wasting power</u>

# Example 7.4: Power-Transfer Efficiency




$$I_{rms} = \frac{300}{|Z|} = 30A$$

$$P = 6 \cdot 30^2 = 5.4kW$$

$$Q = -8 \cdot 30^2 = -7.2kVAr$$

$$Eff = \frac{P_L}{P} = \frac{2}{3} = 67\%, \ P_L = 3.6kW$$

If no capacitor, $I_{rms} = \dfrac{300}{2+20} = 13.6A$

$$\Rightarrow R_s I_{rms}^2 = 372W, \quad RI_{rms}^2 = 3.72kW$$

$$\frac{P_L}{P} = \frac{20}{22} = 91\%$$

# Complex Power

- Apparent power (in VA):

$$V_{rms} I_{rms} = \sqrt{P^2 + Q^2}$$

- Complex power:

$$S \equiv \underline{V}\,\underline{I}^{*} = V_{rms} I_{rms} \angle q = P + jQ$$

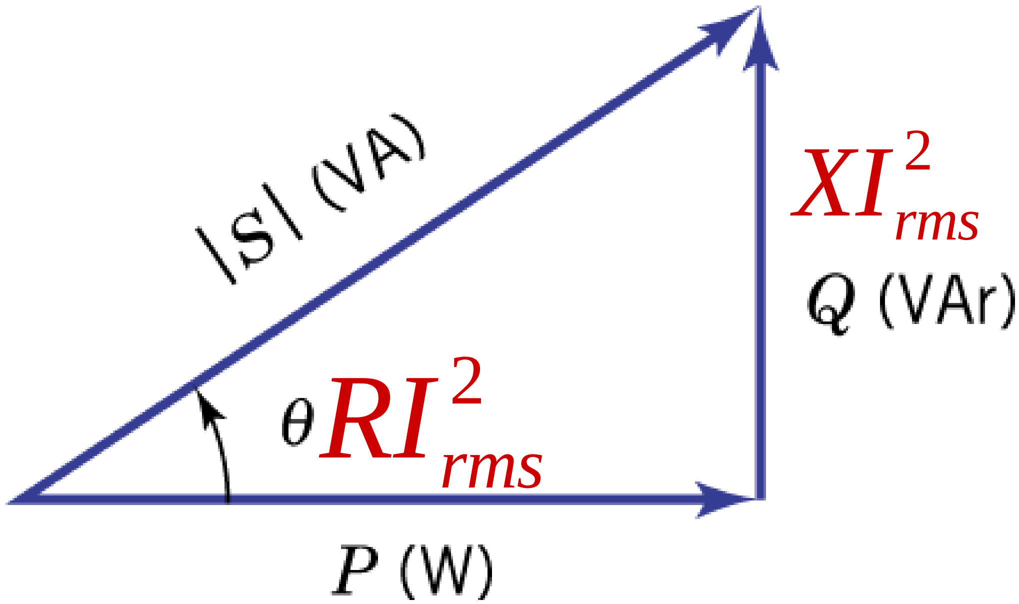
- Magnitude of the complex power = apparent power.

$$|S| = V_{rms} I_{rms}$$

- The complex power obeys the conservation law. In other words, when several loads are connected to the same source, the total complex power from the source is the same as the sum of the complex powers of the loads.

# Power Factor

Power factor: $pf \equiv P/|S| = \cos q$ (for a passive load, $P \geq 0, 0 \leq pf \leq 1$)




$$I_{rms} = \frac{|S|}{V_{rms}} = \frac{P/pf}{V_{rms}} > \frac{P}{V_{rms}}.$$

# Power Factor

- A load with *pf*=1 draws minimum source current.
- If the load is inductive, $Q>0$ and $q>0$, a lagging power factor (current phasor lags the voltage phasor).
- If the load is capacitive, $Q<0$ and $q<0$, a leading power factor (current phasor leads the voltage phasor).
- Connecting a capacitor in parallel with an inductive load can make *pf*=1 (power-factor correction).

$$Q = \pm\sqrt{|S|^2 - P^2} = \pm|S|\sqrt{1 - pf^2}$$

# Table 7.1

**TABLE 7.1 AC Power Quantities**

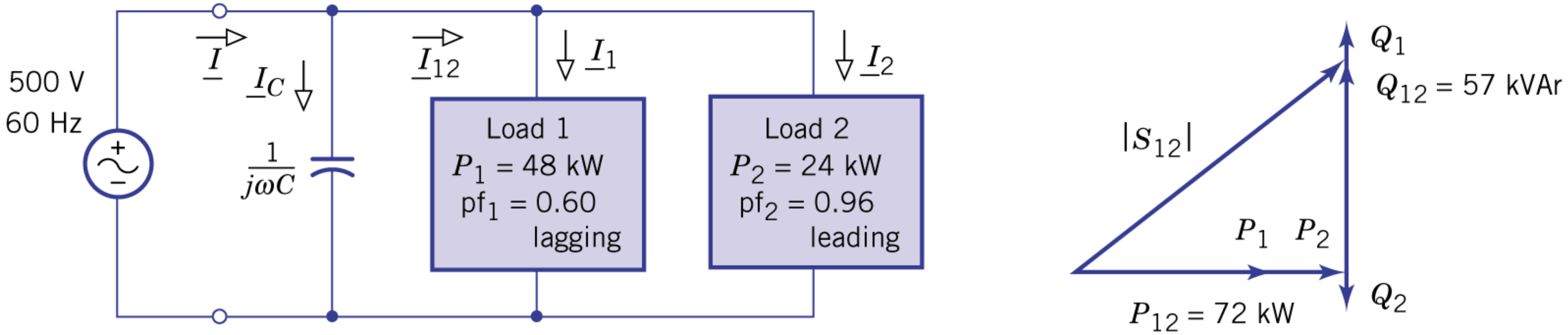
| For load $Z = R + jX = \lvert Z \rvert \angle\theta$ | | | |
|---|---|---|---|
| **Quantity** | **Relations** | **Unit** | **Meaning** |
| Real power | $P = V_{rms} I_{rms} \cos\theta$<br>$= RI_{rms}^2$ | W | Average power delivered to the load |
| Reactive power | $Q = V_{rms} I_{rms} \sin\theta$<br>$= XI_{rms}^2$ | VAr | Rate of reactive energy exchange |
| Complex power | $S = \underline{V}\,\underline{I}^* = P + jQ$<br>$= ZI_{rms}^2$ | | Two-dimensional combination of $P$ and $Q$ |
| Apparent power | $\lvert S \rvert = V_{rms} I_{rms}$<br>$= \sqrt{P^2 + Q^2}$ | VA | Magnitude of complex power |
| Power factor | $\text{pf} = P/\lvert S \rvert$<br>$= \cos\theta$ | | Ratio of real power to apparent power |

# Table 7.2

**TABLE 7.2 Power-Factor Terminology**

| Power Factor | Type of Load | Conditions |
|---|---|---|
| Unity | Resistive | $X = 0,\ \theta = 0,\ Q = 0$ |
| Lagging | Inductive | $X > 0,\ \theta > 0,\ Q > 0$ |
| Leading | Capacitive | $X < 0,\ \theta < 0,\ Q < 0$ |

# Example 7.5: Designing Power-Factor Correction




*(a)* Diagram of industrial plant

*(b)* Power triangle

$$|S_1| = \frac{48k}{0.6} = 80kVA, \quad Q_1 = 64kVAr, |\underline{I}_1| = \frac{80k}{500} = 160A$$

$$|S_2| = \frac{24k}{0.96} = 25kVA, \quad Q_2 = -7kVAr, |\underline{I}_2| = \frac{25k}{500} = 50A$$

$$P_{12} = P_1 + P_2, \quad Q_{12} = Q_1 + Q_2$$

$$|S_{12}| = \sqrt{P_{12}^2 + Q_{12}^2} = 91.8kVA, \quad |I_{12}| = \frac{|S_{12}|}{500} = 184A$$

# Example 7.5: (Cont.)

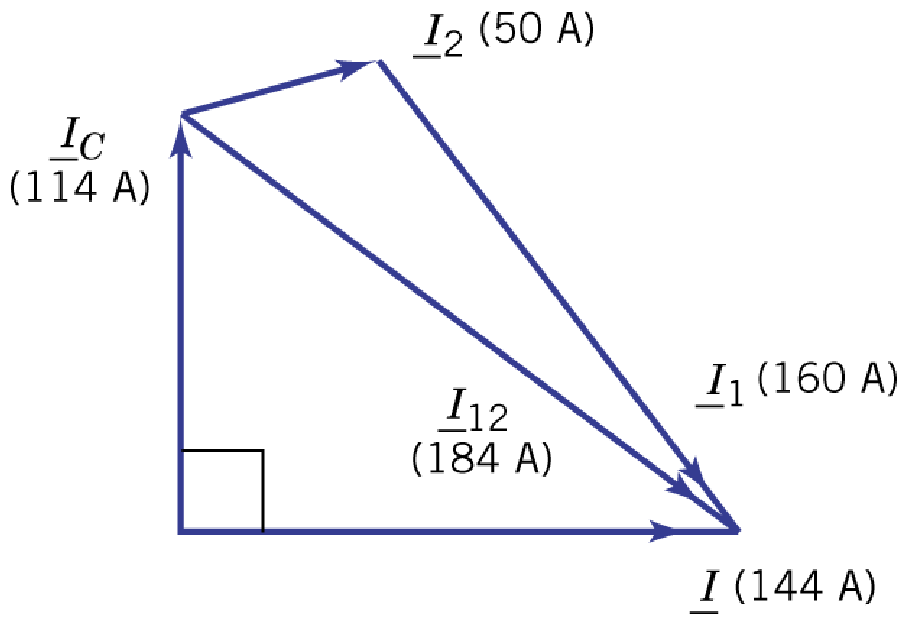
Power factor correction:

$$P_c = 0, \quad Q_c = -57kVAr$$

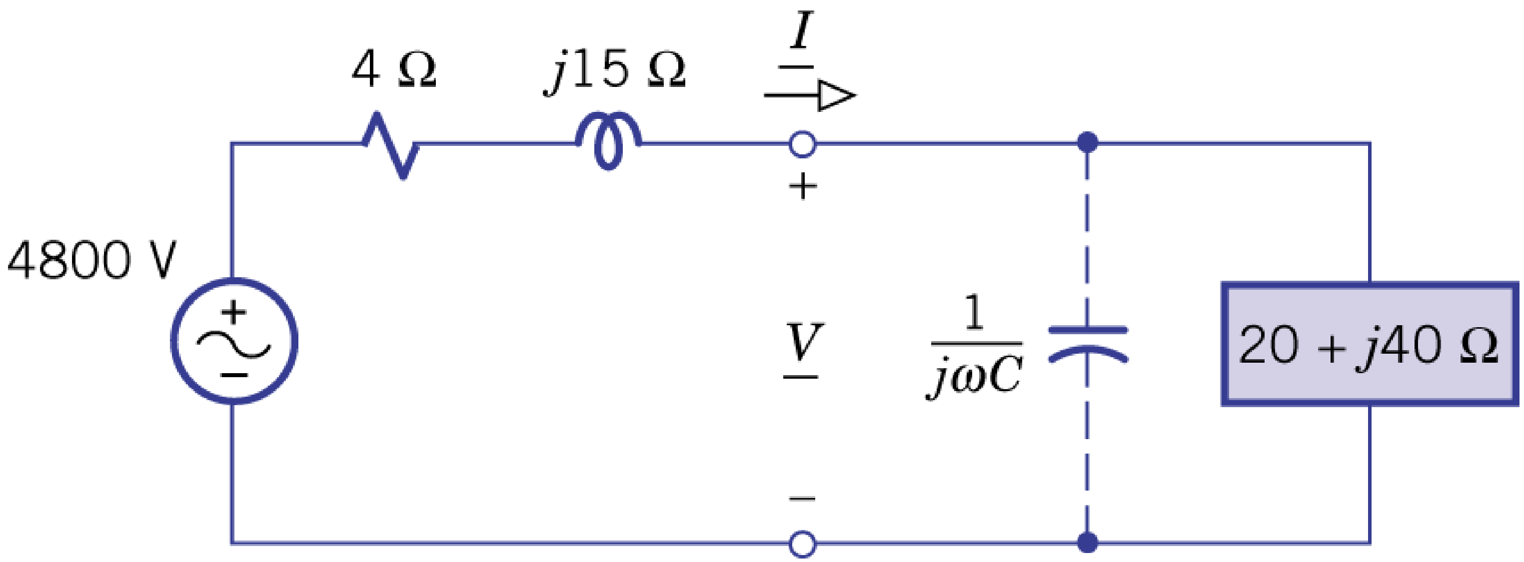
$$C = -\frac{Q_c}{w\left|\underline{V}_c\right|^2} = 605mF$$

$$P = P_{12} + P_c = 72kW, \quad Q = 0$$

$$I_{rms} = \frac{72kW}{500} = 144A$$



# Example 7.6: Improving Power-Transfer Efficiency (only correct the load)




$$I_{rms} = |\underline{I}| = \frac{4800}{|24 + j55|} = 80A$$

$$V_{rms} = |20 + j40| \cdot |\underline{I}| = 3580V$$

$$S = (24 + j55)|\underline{I}|^2 = 154kV + j352kVAr$$

$$\frac{P_L}{P} = \frac{20}{4+20} = 83\%$$

# Example 7.6: (Cont.)

Add $\dfrac{1}{jwC}$

$$Y_{eq} = jwC + \frac{1}{20+j40} = \frac{20}{2000} + j(wC - \frac{40}{2000})$$

after correction $Z_{eq} = \dfrac{2000}{20} = 100$

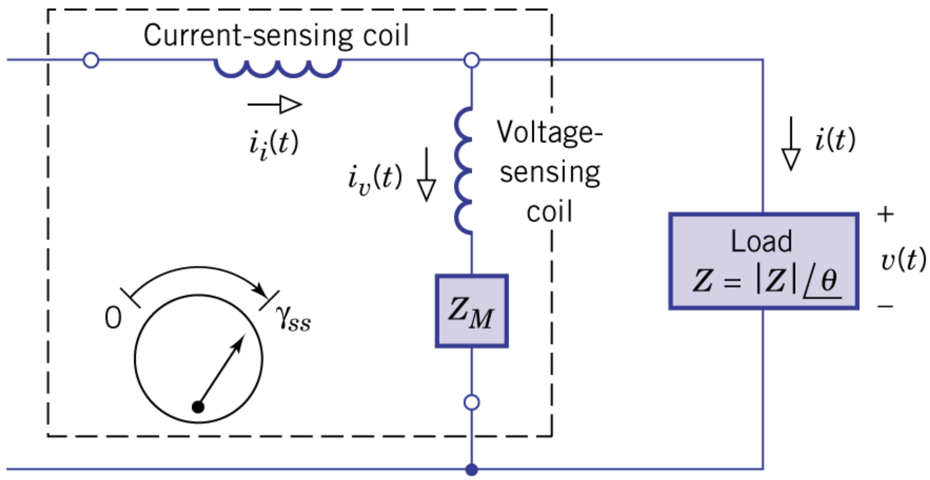
$$100 + (4 + j15) = 104 + j15\Omega$$

$$I_{rms} = \frac{4800}{|104 + j15|} = 45.7A, \quad V_{rms} = 100|\underline{I}| = 4570V$$

$$S = (104 + j15)|\underline{I}|^2 = 217kW + j31kVAr$$

$$\frac{P_L}{P} = \frac{100}{4+100} = 96\%$$

# Wattmeters



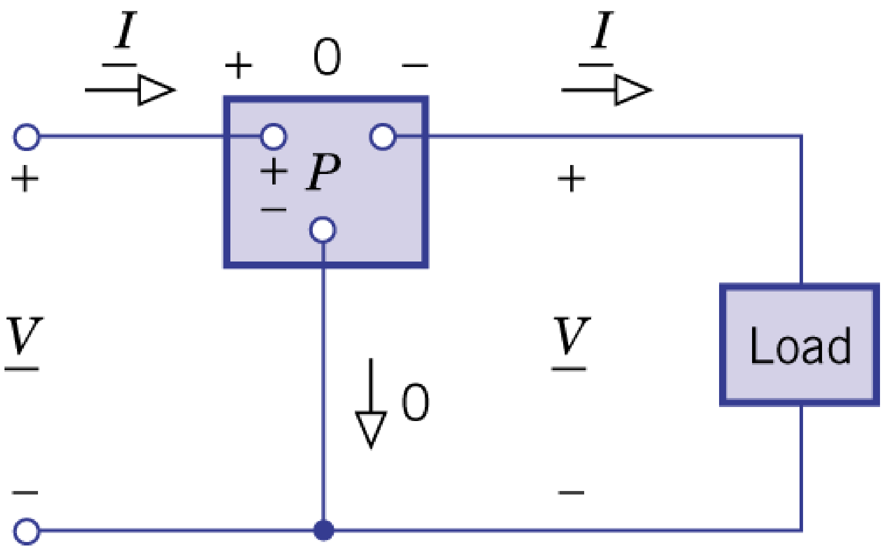

$$\text{Steady state deflection angle } g_{ss} = \frac{K_M}{T}\int_{t_1}^{t_1+T} i_v(t)\, i_i(t)\, dt$$

$$v(t) = \sqrt{2}\left|\underline{V}\right|\cos(wt + f_v), \quad i(t) = \sqrt{2}\left|\underline{I}\right|\cos(wt + f_i)$$

$$\text{Let } R_M >> |Z|$$

$$i_v(t) = \frac{\sqrt{2}\left|\underline{V}\right|}{R_M}\cos\left(wt + f_v\right), \quad i_i(t) \approx \sqrt{2}\left|\underline{I}\right|\cos\left(wt + f_i\right)$$

# Wattmeter $P = \mathrm{Re}\left[\underline{V}\,\underline{I}^*\right]$




$$g_{ss} = \frac{2K_M \left|\underline{V}\right|\left|\underline{I}\right|}{R_M T}\int_{t_1}^{t_1+T} \cos(wt + f_v)\cos(wt + f_i)dt$$

$$= \frac{K_M}{R_M}\left|\underline{V}\right|\left|\underline{I}\right|\cos(f_v - f_i) = \frac{K_M}{R_M}P$$

# Chapter 7: Problem Set

- 4, 7, 12, 16, 19, 22, 26, 31, 34, 40