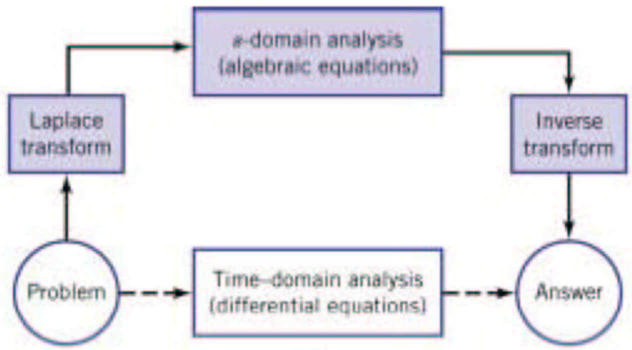
Chapter 13 Laplace Transform Analysis

Chapter 13: Outline

Circuit Complete Response Analysis Using

Laplace Transform



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Laplace Transform and Transform Inversion

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Transform Circuit Analysis

Zero-state response (Natural response and Forced response)

Zero-input response (initial state → independent source)

Complete response = Z.S.R. + Z.I.R.

Laplace transforms

Laplace Transform

- s-domain phasor analysis:

$$x(t) = X_m e^{\sigma t} \cos(\omega t + \phi_x) \longrightarrow \underline{X} = X_m \angle \phi_x$$

- Laplace transform:

$$F(s) = \mathrm{L}[f(t)] \equiv \int_{o^-}^{\infty} f(t) e^{-st} dt$$

$$= \int_{o^+}^{\infty} f(t) e^{-st} dt \quad \text{if } \left| f(t) \right| < \infty, 0^- \le t \le 0^+$$

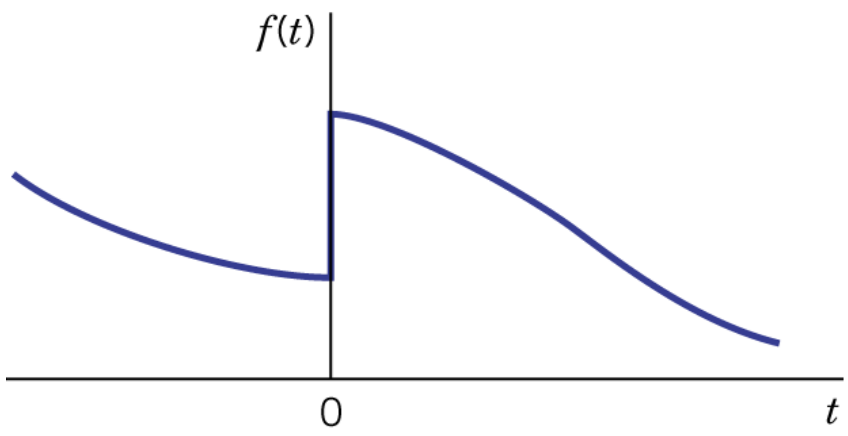
Laplace Transform (Three conditions)

- Unilateral (one-sided Laplace transform): Laplace transform holds for $t=0$. Previous effects are included in the initial conditions at $t=0^-$.
- Existence condition: $\lim_{t\to\infty} |f(t)| e^{-st} = 0, s > s_c$.
- The resulting transform is a function of s.

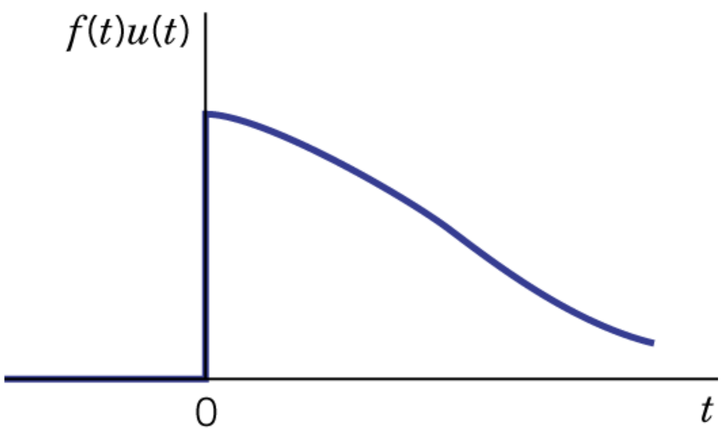
One-Sided Waveform

$$\text{unit step}: u(t) = \begin{cases} 0, t < 0 \\ 1, t > 0 \end{cases} \Rightarrow f(t)u(t) = \begin{cases} 0, t < 0 \\ f(t), t > 0 \end{cases}$$

$$\Rightarrow L\left[f(t)u(t) \right] = L\left[f(t) \right], \text{ if } \left| f(t) \right| < \infty, 0^- \le t \le 0^+$$



(a) Waveform with discontinuity at $t = 0$



(b) One–sided waveform

Sufficient Existence Condition

$$\int_{0^-}^{\infty} f(t)e^{-st}dt \text{ is finite}$$

$$\int_{0^-}^{\infty} f(t)e^{-st}dt \le \int_{0^-}^{\infty} \left| f(t)e^{-st} \right| dt = \int_{0^-}^{\infty} \left| f(t) \right| \left| e^{-st} \right| dt = \int_{0^-}^{\infty} \left| f(t) \right| e^{-st} dt$$

$$\Rightarrow \text{The integral remains finite if } \lim_{t \to \infty} \int_{0^-}^{\infty} \left| f(t) \right| e^{-st} dt = 0, s > s_c$$

$$\Rightarrow f(t) \text{ is of exponential order}$$

$$\Rightarrow s_c : \text{abscissa of convergence}$$

Inverse Laplace Transform

- Inverse Laplace transform for *t=0*:

$$f(t) = \mathrm{L}^{-1}[F(s)] \equiv \frac{1}{2pj} \int_{c-jw}^{c+jw} F(s) e^{st} ds, c > s_{c}$$

- Inverse Laplace transform is usually done by partial fraction expansion (will be covered in section 13.2).

Example 13.1

$$L\left[e^{-at}\right]$$

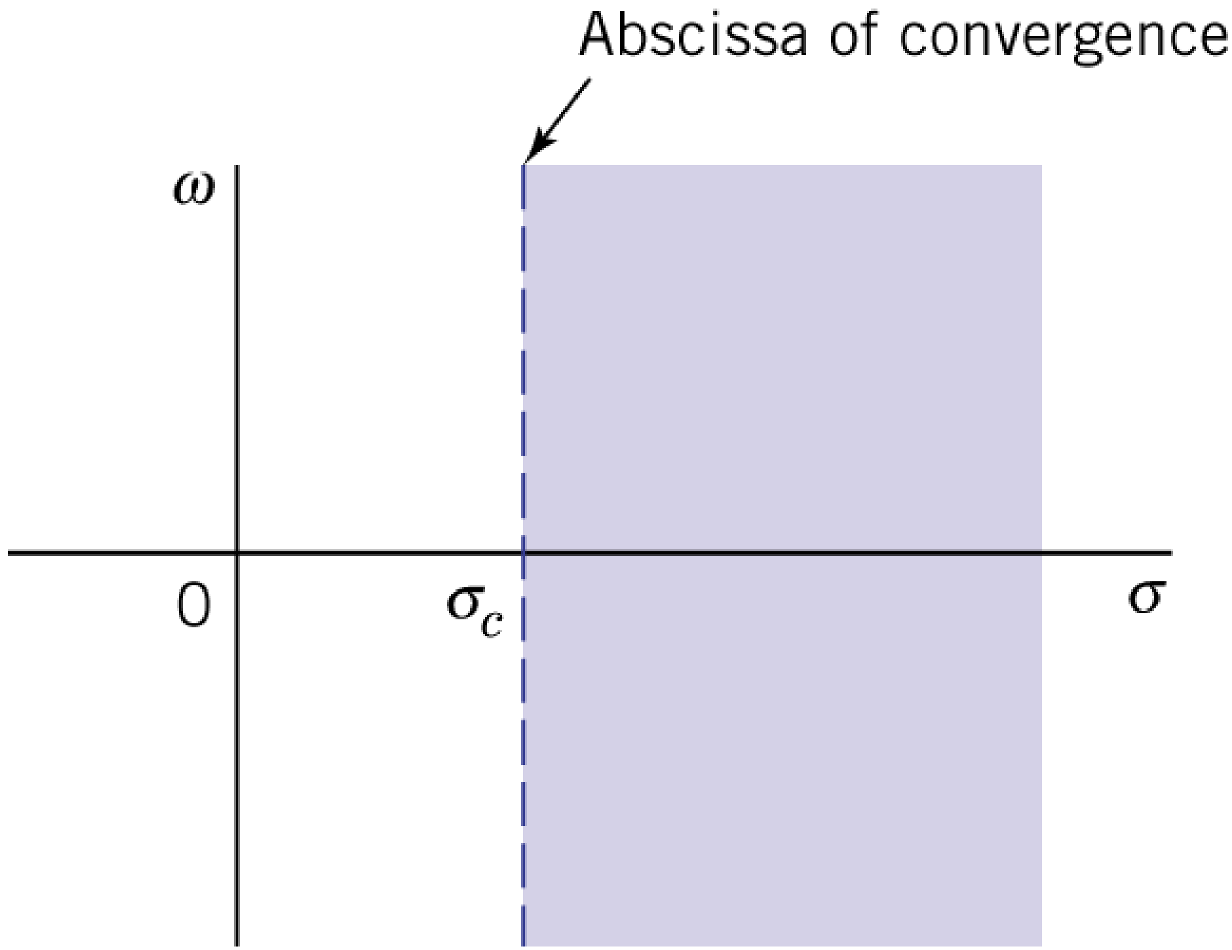
$$\lim_{t\to\infty}\left|f(t)\right|e^{-at} = \lim_{t\to\infty} e^{-(a+s)t} = 0, \text{ if } a+s > 0$$

$$L\left[e^{-at}\right] = \int_{0^-}^{\infty} e^{-at}e^{-st}dt = \int_{0^-}^{\infty} e^{-(a+s)t}dt = \frac{1}{s+a}, \text{ when } s_c > -a$$

$$\text{if } a = 0,\ L[1] = L[u(t)] = \frac{1}{s}$$

$$(\text{Note}: a \text{ can be complex, } \operatorname{Re}[a] + s > 0)$$

Example 13.1



Example 13.2

$$L[\sin bt] = L\left[\frac{1}{j2}\left(e^{jbt} - e^{-jbt}\right)\right]$$

$$= \frac{1}{j2}\left(\frac{1}{s - jb} - \frac{1}{s + jb}\right) = \frac{b}{s^2 + b^2}$$

$$s_c = 0$$

Transform Properties

– Linear combination:

$$L\left[f(t)\right] = F(s),\ L\left[g(t)\right] = G(s)$$

$$L\left[Af(t) + Bg(t)\right] = AF(s) + BG(s)$$

$$L\left[Af(t)\right] = AF(s)$$

Transform Properties

– Multiplication by e^{-at}:

$$L\left[f(t)\right] = F(s)$$

$$L\left[e^{-at} f(t)\right] = F(s+a)$$

Transform Properties

– Multiplication by *t*:

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_{0^-}^{\infty} f(t)e^{-st}dt$$

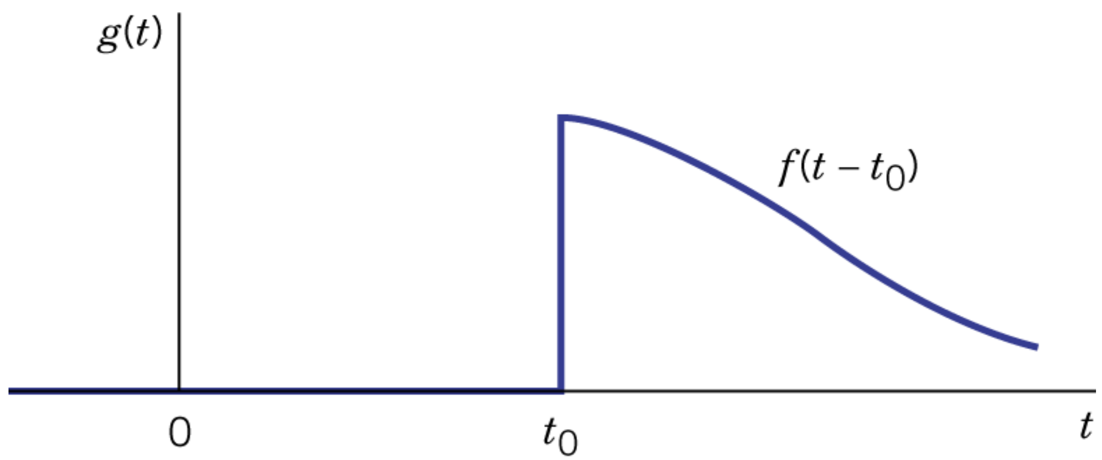
$$= -\int_{0^-}^{\infty} tf(t)e^{-st}dt = -L[tf(t)]$$

Transform Properties

– Time delay:

$$g(t) = f(t - t_0)u(t - t_0) = \begin{cases} 0, t < t_0 \\ f(t - t_0), t > t_0 \end{cases}$$

$$L[g(t)] = e^{-st_0} F(s)$$



Transform Properties

– Differentiation:

$$L\left[f'(t)\right] = L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^-)$$

$$L\left[f''(t)\right] = s^2F(s) - sf(0^-) - f'(0^-)$$

Transform Properties

– Integration:

$$g(t) = \int_{0^-}^{t} f(l)\,dl$$

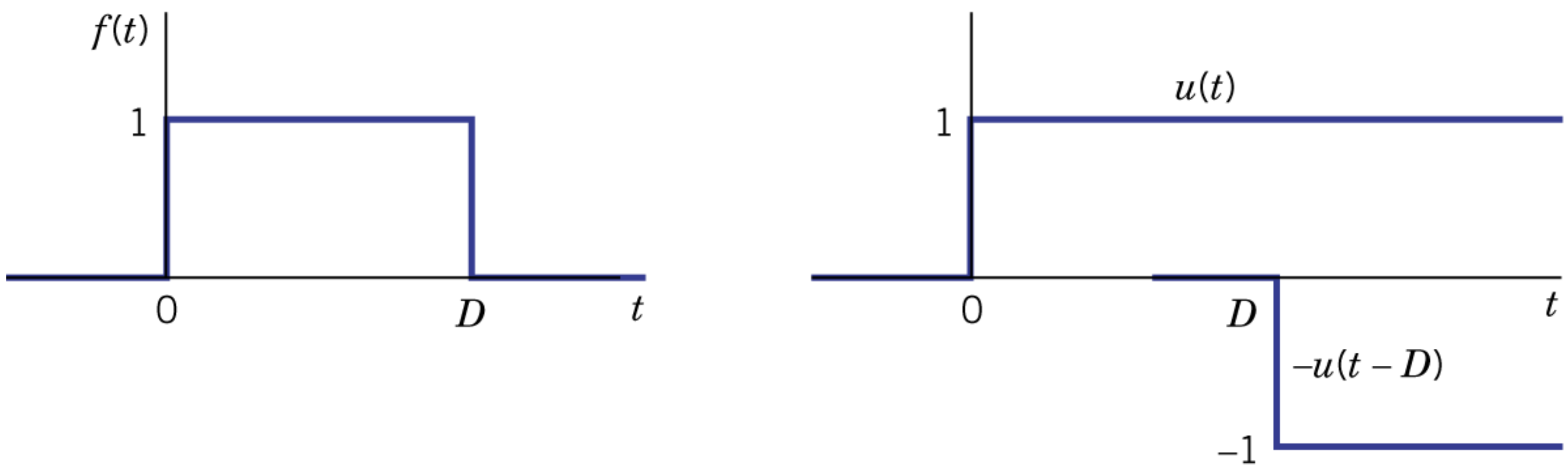
$$dg(t) = f(t)dt,\ \ g(0^-) = 0,\ \ e^{-st}dt = d\left(\frac{e^{-st}}{-s}\right)$$

$$L[g(t)] = L\left[\int_{0^-}^{t} f(l)\,dl\right] = \frac{1}{s}F(s)$$

Example 13.3

$$f(t) = \begin{cases} 1, & 0 < t < D \\ 0, & \text{otherwise} \end{cases} = u(t) - u(t-D)$$

$$F(s) = \frac{1}{s} + (-1)e^{-sD}\frac{1}{s} = \frac{1 - e^{-sD}}{s}$$



(a) Rectangular pulse (b) Decomposition as two step functions

Example 13.4

$$f(t) = \sin bt \Rightarrow L[f(t)] = \frac{b}{s^2 + b^2}$$

$$L[f'(t)] = L[b\cos bt] = \frac{sb}{s^2 + b^2} - \sin(0^-) = \frac{sb}{s^2 + b^2}$$

$$L[\cos bt] = \frac{s}{s^2 + b^2}$$

$$L[\cos(bt + f)] = L[\cos f \cos bt - \sin f \sin bt] = \frac{s\cos f - b\sin f}{s^2 + b^2}$$

Table 13.1

TABLE 13.1 Laplace Transform Properties

Operation	Time Function	Laplace Transform
Linear combination	$Af(t) + Bg(t)$	$AF(s) + BG(s)$
Multiplication by e^{-at}	$e^{-at}f(t)$	$F(s + a)$
Multiplication by t	$tf(t)$	$-dF(s)/ds$
Time delay	$f(t - t_0)u(t - t_0)$	$e^{-st_0}F(s)$
Differentiation	$f'(t)$	$sF(s) - f(0^-)$
	$f''(t)$	$s^2F(s) - sf(0^-) - f'(0^-)$
Integration	$\int_{0^-}^{t} f(\lambda)\, d\lambda$	$\frac{1}{s} F(s)$

Table 13.2

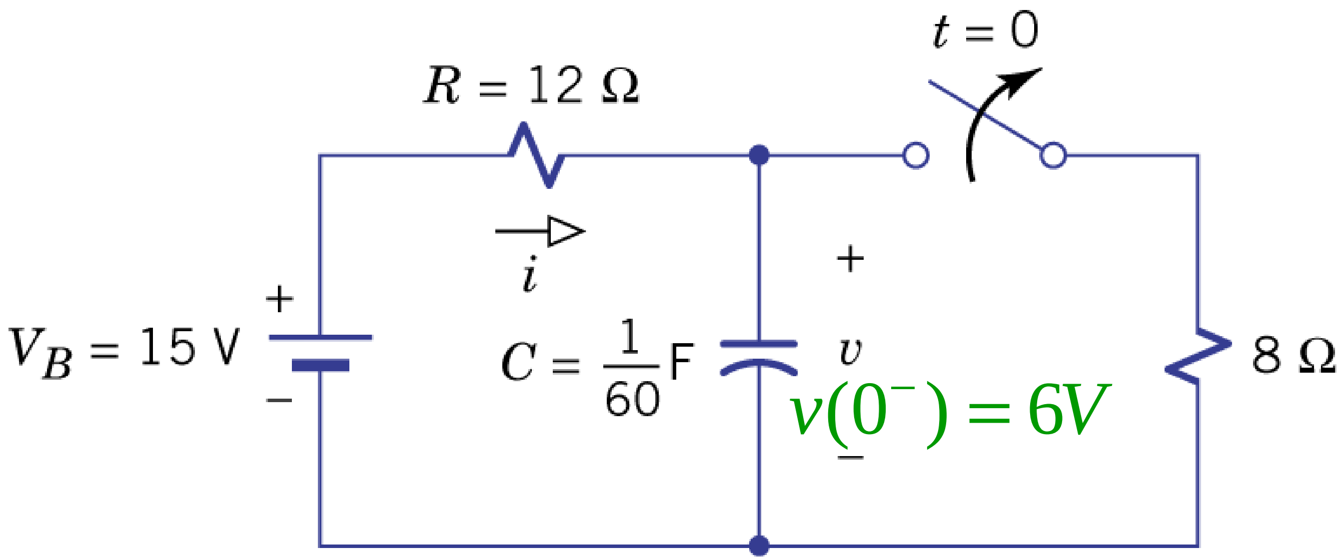
TABLE 13.2 Laplace Transform Pairs

$f(t)$	$F(s)$
A	$\dfrac{A}{s}$
$u(t) - u(t - D)$	$\dfrac{1 - e^{-sD}}{s}$
t	$\dfrac{1}{s^2}$
t^r	$\dfrac{r!}{s^{r+1}}$
e^{-at}	$\dfrac{1}{s + a}$
te^{-at}	$\dfrac{1}{(s + a)^2}$
$t^r e^{-at}$	$\dfrac{r!}{(s + a)^{r+1}}$
$\sin \beta t$	$\dfrac{\beta}{s^2 + \beta^2}$
$\cos (\beta t + \phi)$	$\dfrac{s \cos \phi - \beta \sin \phi}{s^2 + \beta^2}$
$e^{-at} \cos (\beta t + \phi)$	$\dfrac{(s + a) \cos \phi - \beta \sin \phi}{(s + a)^2 + \beta^2}$

Solving Differential Equations

- The transformation automatically incorporates the initial conditions (zero-input response).
- Transformation converts linear differential equations to *s*-domain algebraic equations.
- Transformation is **similar** to the *s*-domain phasor analysis. Denominator of the *s*-domain function includes the characteristic polynomial.
- Inverse transformation is required to obtain the resultant time domain function.

First-Order Example



$$0.2v'(t) + v(t) = 15$$

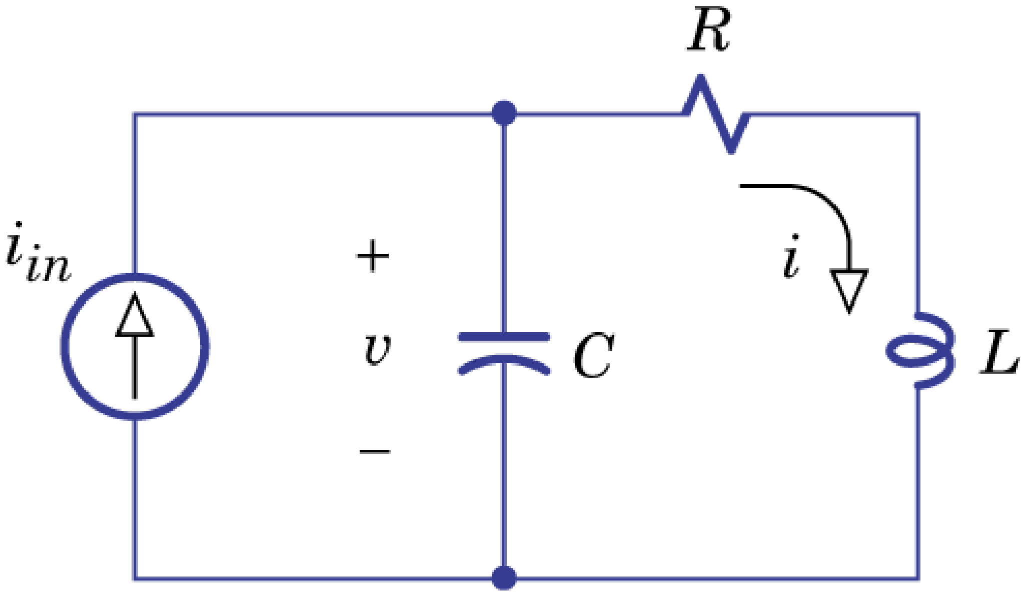
$$0.2\left[sV(s) - v(0^-)\right] + V(s) = \frac{15}{s}$$

$$V(s) = \frac{6s+75}{s(s+5)} = \frac{6}{s+5} + \frac{75}{s(s+5)}$$

$$v(t) = L^{-1}\left[V(s)\right] = L^{-1}\left[\frac{6}{s+5}\right] + L^{-1}\left[\frac{75}{s(s+5)}\right]$$

$$= 6e^{-5t} + 75(-0.2)\left(e^{-5t} - 1\right) = 15 - 9e^{-5t}, t \geq 0$$

Second-Order Example



$$i_{in} = \begin{cases} I_1, & t < 0 \\ I_2, & t > 0 \end{cases}$$

$$i(0^-) = I_1, v(0^-) = RI_1$$

Second Order Example (Cont.)

$t > 0$

$$\begin{cases} Li'(t) + Ri(t) = v(t) \\ Cv'(t) + I(t) = I_2 \end{cases} \Rightarrow \begin{cases} L\left[sI(s) - I_1\right] + RI(s) = V(s) \\ C\left[sV(s) - RI_1\right] + I(s) = I_2 / s \end{cases}$$

$$I(s) = \frac{I_1 s^2 + (R/L)I_1 s + I_2 / LC}{s\left[s^2 + (R/L)s + 1/LC\right]}$$

$\boxed{\text{Characteristic polynomial}}$

if $I_2 = I_1$ (no switching)

$$I(s) = \frac{I_1}{s} \Rightarrow i(t) = I_1, \text{ for } t \geq 0$$

Transform Inversion

Partial-Fraction Expansion

- Partial-fraction expansion of a strictly proper rational function:

$$F(s)=\frac{N(s)}{D(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0},\, m\le n-1$$

- Three cases will be considered: distinct real poles, complex poles and repeated poles.

Case 1: Distinct Real Poles

- (Heaviside's theorem, cover-up rule.)

$$F(s) = \frac{N(s)}{D(s)} = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \cdots + \frac{A_n}{s - s_n}$$

$$f(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \cdots + A_n e^{s_n t}, \ t \geq 0$$

$$A_i = (s - s_i) F(s) \Big|_{s = s_i}, \quad i = 1, 2, \ldots$$

Example 13.5: Inversion of a Third-Order Function (Heaviside's theorem)

$$I(s) = \frac{I_1 s^2 + (R/L)I_1 s + I_2/LC}{s\left[s^2 + (R/L)s + 1/LC\right]} \qquad \begin{pmatrix} R = 12\Omega, L = 1H, C = \dfrac{1}{20}F \\ I_1 = -2A, I_2 = 2A \end{pmatrix}$$

$$I(s) = \frac{-2s^2 - 24s + 40}{s(s+2)(s+10)} = \frac{A_1}{s} + \frac{A_2}{s+2} + \frac{A_3}{s+10}$$

$$A_1 = sI(s)\Big|_{s=0} = 2$$

$$A_2 = (s+2)I(s)\Big|_{s=-2} = -5$$

$$A_3 = (s+10)I(s)\Big|_{s=-10} = 1$$

$$I(s) = \frac{2}{s} + \frac{-5}{s+2} + \frac{1}{s+10} \Rightarrow i(t) = 2 - 5e^{-2t} + e^{-10t}, t \geq 0$$

Example 13.5: (Method of Undetermined Coefficients)

$$I(s) = \frac{-2s^2 - 24s + 40}{s(s+2)(s+10)} = \frac{2}{s} - \frac{5}{s+2} + \frac{A_3}{s+10}$$

$$I(s)\bigg|_{s=1} = \frac{14}{33} = 2 - \frac{5}{3} + \frac{A_3}{11} = \frac{11+3A_3}{33}$$

$$\Rightarrow A_3 = 1$$

Case 2: Complex Poles

$$D(s) = (s^2 + 2\alpha s + \omega_0^2)(s - s_3)\cdots(s - s_n)$$

$$\text{if } a^2 < w_0^2,\ s_1, s_2 = -a \pm jb = -a \pm j\sqrt{w_0^2 - a^2}$$

$$F(s) = G(s) + \sum_{i=3}^{n} \frac{A_i}{s - s_i}$$

$$G(s) = \frac{\frac{1}{2}K}{s + a - jb} + \frac{\frac{1}{2}K^*}{s + a + jb}$$

$$K = 2(s + a - jb)F(s)\Big|_{s=-a+jb} = K_m e^{jf}$$

$$g(t) = \frac{1}{2}Ke^{-(a-jb)t} + \frac{1}{2}K^* e^{-(a+jb)t} = K_m e^{-at}\cos(bt + f)$$

Case 2: Complex Poles (Cont.)

Or,...

Undetermined coefficients

$$g(t) = K_m e^{-at}\cos(bt + f)$$

$$L[g(t)] = \frac{(s+a)K_m \cos f - bK_m \sin f}{(s+a)^2 + b^2}$$

$$G(s) = \frac{Bs + C}{s^2 + 2as + w_0^2} = \frac{(s+a)B - (aB - C)}{(s+a)^2 + w_0^2 - a^2}$$

$\Rightarrow$ Find a, b, w_0^2, B and C, compare coefficients, then determine K.

$$\Rightarrow B + j(aB - C)/b = K_m \cos f + jK_m \sin f = K$$

Example 13.6: Inversion with Complex Poles

$$F(s)=\frac{15s^2-16s-7}{(s+2)(s^2+6s+25)} \Rightarrow s_1=-2, \quad s_2,s_3=-3\pm j4=-a\pm jb$$

$$F(s)=\frac{A_1}{s+2}+G(s)$$

$$G(s)=\frac{\frac{1}{2}K}{s+3-j4}+\frac{\frac{1}{2}K^*}{s+3+j4}=\frac{Bs+C}{s^2+6s+25}$$

$$A_1=(s+2)F(s)\Big|_{s=-2}=5$$

$$K=2\frac{15s^2-16s-7}{(s+2)(s^2+6s+25)}\Bigg|_{s=-3+j4}$$

Example 13.6: (Cont., method of undetermined coefficients)

$$F(s)=\frac{15s^2-16s-7}{(s+2)(s^2+6s+25)}=\frac{5}{s+2}+\frac{Bs+C}{s^2+6s+25}$$

$$\Rightarrow sF(s)=\frac{15s^3+\cdots}{s^3+\cdots}=\frac{5s}{s+2}+\frac{Bs^2+Cs}{s^2+6s+25}$$

$$s\to\infty,\ 15=5+B\Rightarrow B=10$$

$$F(s)=\frac{15s^2-16s-7}{(s+2)(s^2+6s+25)}=\frac{5}{s+2}+\frac{10s+C}{s^2+6s+25}$$

$$F(0)=\frac{-7}{2\cdot 25}=\frac{5}{2}+\frac{C}{25}\Rightarrow C=-66$$

$$g(t)=K_m e^{-at}\cos(bt+f)$$

$$K=B+j\frac{aB-C}{b}=10+j24=26\angle 67.4^0$$

$$f(t)=5e^{-2t}+g(t)=5e^{-2t}+26e^{-3t}\cos(4t+67.4^0),\quad t\ge 0$$

Case 3: Repeated Poles

$$F(s) = G(s) + \frac{A_3}{s - s_3} + \cdots + \frac{A_n}{s - s_n}$$

$$G(s) = \frac{A_{i1}}{s - s_i} + \frac{A_{i2}}{(s - s_i)^2} \quad \text{(double poles)}$$

$$g(t) = A_{i1} e^{s_i t} + A_{i2} t e^{s_i t}$$

$$(s - s_i)^2 F(s) = (s - s_i) A_{i1} + A_{i2} + (s - s_i)^2 \sum_{j=3}^{n} \frac{A_j}{s - s_j}$$

$$\Rightarrow A_{i2} = (s - s_i)^2 F(s) \Big|_{s = s_i}$$

$$A_{i1} = \frac{d}{ds} \left[(s - s_i)^2 F(s) \right] \Big|_{s = s_i}$$

Case 3: Repeated Poles (Cont.)

$$G(s) = \frac{A_{i1}}{s - s_i} + \frac{A_{i2}}{(s - s_i)^2} + \frac{A_{i3}}{(s - s_i)^3}$$

$$g(t) = A_{i1} e^{s_i t} + A_{i2} t e^{s_i t} + \frac{1}{2} A_{i3} t^2 e^{s_i t}$$

$$\Rightarrow A_{i3} = \left[(s - s_i)^3 F(s) \right] \Big|_{s = s_i}$$

$$A_{i2} = \frac{d}{ds} \left[(s - s_i)^3 F(s) \right] \Big|_{s = s_i}$$

$$A_{i1} = \frac{1}{2!} \frac{d^2}{ds^2} \left[(s - s_i)^3 F(s) \right] \Big|_{s = s_i}$$

Example 13.7: Inversion with a Triple Pole

$$F(s)=\frac{-s^2-2s+14}{(s+4)^3(s+5)}=\frac{A_{11}}{s+4}+\frac{A_{12}}{(s+4)^2}+\frac{A_{13}}{(s+4)^3}+\frac{A_4}{s+5}$$

$$A_4=(s+5)F(s)\Big|_{s=-5}=1,\quad A_{13}=(s+4)^3F(s)\Big|_{s=-4}=6$$

$$F(s)=\frac{-s^2-2s+14}{(s+4)^3(s+5)}=\frac{A_{11}}{s+4}+\frac{A_{12}}{(s+4)^2}+\frac{6}{(s+4)^3}+\frac{1}{s+5}$$

$$\lim_{s\to\infty}sF(s)=0\Rightarrow A_{11}=-1$$

$$F(0)=\frac{14}{320}=-\frac{1}{4}+\frac{A_{12}}{16}+\frac{6}{64}+\frac{1}{5}\Rightarrow A_{12}=0$$

$$F(s)=-\frac{1}{s+4}+\frac{0}{(s+4)^2}+\frac{6}{(s+4)^3}+\frac{1}{s+5}$$

$$f(t)=-e^{-4t}+\frac{6}{2}t^2e^{-4t}+e^{-5t},\ t\geq 0$$

Time Delay

Initial-Value Theorem

Final-Value Theorem

Time Delay

$$L\left[g(t-t_0)u(t-t_0)\right] = L\left[g(t)\right]e^{-st_0}$$

$$F(s) = \frac{N_1(s) + N_2(s)e^{-st_0}}{D(s)} = F_1(s) + F_2(s)e^{-st_0}$$

$$f(t) = f_1(t) + f_2(t-t_0)u(t-t_0), \quad t \geq 0$$

Example 13.8: Inversion with Time Delay

$$\text{excitation}: x(t) = 20u(t) - 40u(t-3)$$

$$y'(t) - 5y(t) = -x(t) = -20u(t) + 40u(t-3), \;\; y(0^-) = 0$$

$$sY(s) - 5Y(s) = \frac{-20 + 40e^{-3s}}{s}$$

$$Y(s) = \frac{-20 + 40e^{-3t}}{s(s-5)} = F_1(s) - 2F_1(s)e^{-3s}$$

$$F_1(s) = \frac{-20}{s(s-5)} = \frac{4}{s} + \frac{-4}{s-5}$$

$$f_1(t) = 4 - 4e^{5t}$$

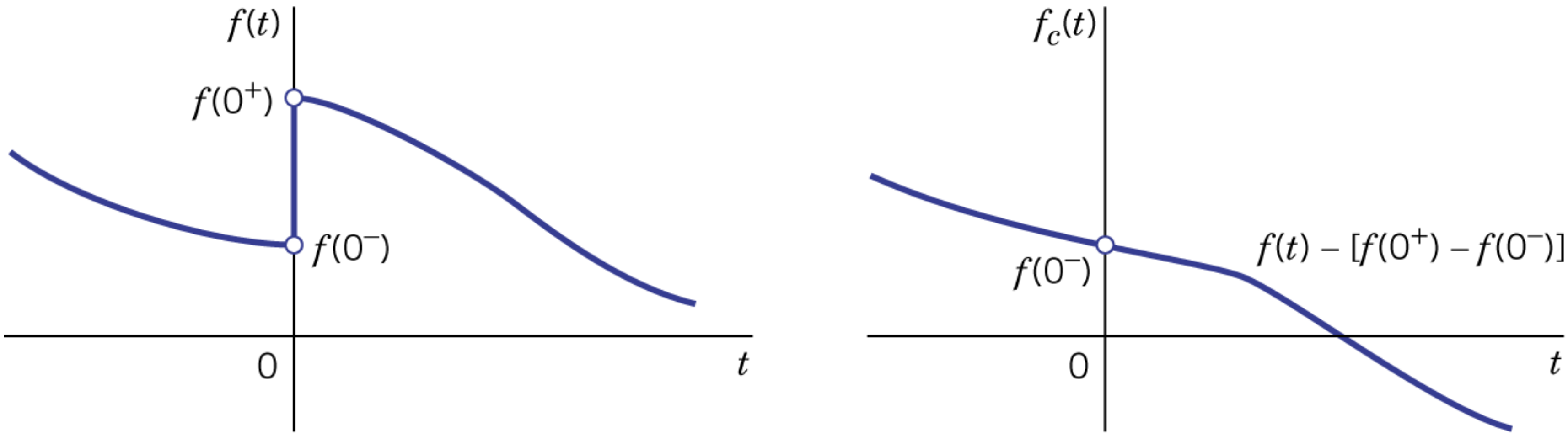
$$y(t) = f_1(t) - 2f_1(t-3)u(t-3) = 4 - 4e^{5t} - \left[8 - 8e^{5(t-3)}\right]u(t-3), \;\; t \geq 0$$

Initial-Value Theorem

$$f(0^+) = \lim_{s\to\infty} sF(s)$$

$$\left(F(s)\ \text{is strictly proper}\right)$$

Initial-Value Theorem (Cont.)



(a) Waveform with jump at $t = 0$ *(b)* Continuous waveform

$$f_c(t) = f(t) - \left[f(0^+) - f(0^-) \right] u(t)$$

$$\left(\text{i.e., } f_c(t) - f(0^-) = f(t) - f(0^+), \ t > 0 \right)$$

Initial-Value Theorem (Cont.)

$$F_c(s) = F(s) - \left[f(0^+) - f(0^-) \right] / s$$

$$L\left[f_c'(t) \right] = sF_c(s) - f_c(0^-) = \int_{0^-}^{\infty} f_c'(t) e^{-st} dt$$

$$\lim_{s \to \infty} \int_{0^-}^{\infty} f_c'(t) e^{-st} dt = 0 = \lim_{s \to \infty} sF_c(s) - f_c(0^-)$$

$$\left(\left| f_c'(t) \right| \text{is finite over } 0^- \leq t \leq 0^+ \right)$$

$$\Rightarrow sF(s) - \left[f(0^+) - f(0^-) \right] - f(0^-) = 0$$

$$\lim_{s \to \infty} sF(s) - f(0^+) = 0$$

Initial Slope

Find initial slope $f'(0^+)$ assuming $f(0^-)$ is known

$$L[f'(t)] = sF(s) - f(0^-) = \frac{sN(s)}{D(s)} - f(0^-)$$

$$f'(0^+) = \lim_{s\to\infty} s\,\frac{sN(s) - f(0^-)D(s)}{D(s)}$$

Final-Value Theorem

$$f(\infty) = \lim_{s \to 0} sF(s)$$

$$\left(\begin{array}{c} \text{no poles in the RHP,} \\ \text{no multiple poles at the origin,} \\ \text{no poles on the imaginary axis} \end{array} \right)$$

$$L\left[f'(t)\right] = sF(s) - f(0^-) = \int_{0^-}^{\infty} f'(t) e^{-st} dt$$

$$\lim_{s \to 0} sF(s) - f(0^-) = \int_{0^-}^{\infty} f'(t) dt = f(\infty) - f(0^-)$$

Example 13.9: Calculating Initial and Final Values

$$f(0^-) = 5$$

$$F(s) = \frac{N(s)}{D(s)} = \frac{5s^3 - 1600}{s(s^3 + 18s^2 + 90s + 800)} \text{ (strictly proper)}$$

$$f(0^+) = \lim_{s \to \infty} sF(s) = 5 = f(0^-)$$

$$f'(0^+) : \frac{sN(s) - f(0^-)D(s)}{D(s)} = \frac{-90s^3 - 450s^2 - 5600s}{s(s^3 + 18s^2 + 90s + 800)} \text{ (strictly proper)}$$

$$f'(0^+) = \lim_{s \to \infty} s \frac{-90s^3 - \cdots}{s^4 + \cdots} = -90$$

Example 13.9: (Cont.)

$$f(0^-) = 5$$

$$F(s) = \frac{N(s)}{D(s)} = \frac{5s^3 - 1600}{s(s^3 + 18s^2 + 90s + 800)} \text{ (strictly proper)}$$

For final values, check pole locations

$$f(\infty) = \lim_{s \to 0} sF(s) = -2$$

Transform circuit analysis

Transform Circuit Analysis

- Given a circuit with some initial state at $t=0^-$ and an excitation $x(t)$ starting at $t=0$, find the resulting behavior of any voltage or current $y(t)$ for $t=0$.
- Zero-state response, natural response, forced response, zero-input response and complete response.

Zero-State Response

- Zero-state response: a circuit with no stored energy at $t=0^-$.

$$\text{For an n-th order network,}$$

$$a_n \frac{d^n y}{dt^n} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + \cdots + b_1 \frac{dx}{dt} + b_0 x$$

$$\Downarrow$$

$$Y(s) = L[y(t)], Y(s) = L[y(t)]$$

$$L\left[\frac{d^k y}{dt^k}\right] = s^k Y(s) \;\; (\text{"zero initial state"})$$

$$\Downarrow$$

$$H(s) \equiv \frac{\underline{Y}}{\underline{X}} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$\left(\text{same as the } s\text{-domain network function}\right)$$

Zero-State Response

Steps:

(1). Draw s-domain diagram, obtain $H(s)=\dfrac{Y(s)}{X(s)}$

(2). $x(t)\rightarrow X(s),\quad Y(s)=H(s)X(s)$

(3). $y(t)=L^{-1}\left[Y(s)\right]$

$$H(s)=\frac{N_H(s)}{P(s)},\quad X(s)=\frac{N_X(s)}{D_X(s)}$$

$P(s)$: characteristic polynomial

Step Response
(*zero initial state, by definition*)

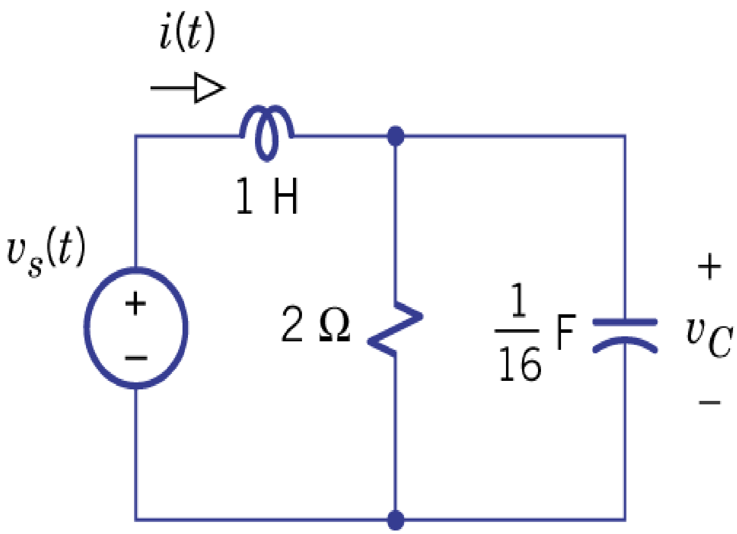
$$x(t) = u(t) = 1, t > 0$$

$$X(s) = 1/s$$

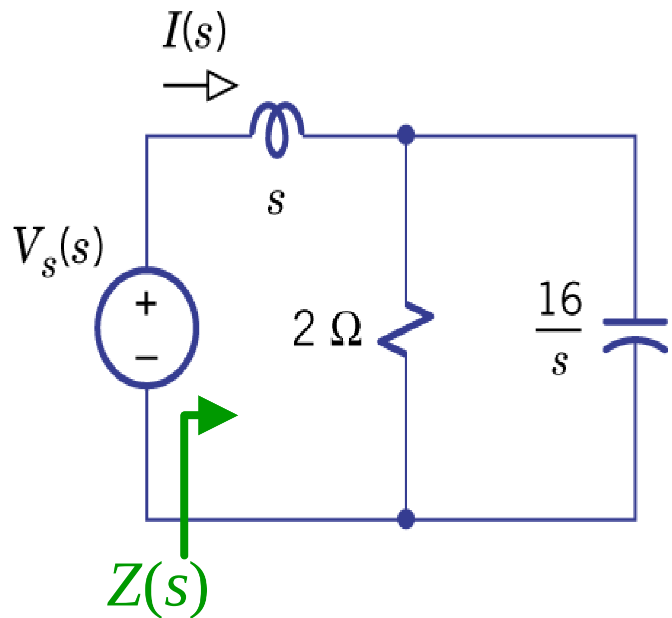
$$Y(s) = H(s) \times \frac{1}{s} = \frac{N_H(s)}{sP(s)}$$

A pole at the origin

Example 13.10: Step Response



(a) Circuit for Example 13.10



(b) *s*-domain diagram

$$I(s) = V_s(s)H(s)$$

$$\frac{1}{H(s)} = Z(s) = s + (2 \| s/16), \quad V_s(s) = \frac{1}{s}$$

Example 13.10: (Cont.)

$$I(s) = H(s) \times \frac{1}{s} = \frac{s+8}{s(s+4)^2} = \frac{0.5}{s} - \frac{0.5}{s+4} - \frac{1}{(s+4)^2}$$

$$i(t) = 0.5 - 0.5e^{-4t} - te^{-4t}, \quad t \geq 0$$

Steady state response

natural response

$$i(0^+) = \lim_{s \to \infty} sI(s) = 0$$

$$i(\infty) = \lim_{s \to 0} sI(s) = 0.5$$

Zero-state AC response

<u>Forced response</u>
<u>From phasor analysis</u>

$$x(t) = X_m \cos(bt + f_x)$$

$$D_X(s) = s^2 + b^2$$

Natural response
from $Y(s)$

$$Y(s) = \frac{N_H(s)N_X(s)}{P(s)(s^2 + b^2)} \equiv Y_N(s) + Y_F(s) = \frac{N_N(s)}{P(s)} + \frac{N_F(s)}{s^2 + b^2}$$

$$y(t) = y_N(t) + y_F(t)$$

$$y_F(t) = Y_m \cos(bt + f_y) \quad \text{where } \underline{Y} = H(jb)\underline{X} = Y_m \angle f_y$$

Example 13.11: Zero-State AC Response (from Fig. 13.9)

$$v_s(t) = 50\cos 8t,\ t \geq 0 \Rightarrow V_s(s) = \frac{50s}{s^2+64}$$

$$I(s) = H(s)V_s(s) = \frac{50s^2+400s}{(s+4)^2(s^2+64)}$$

$$= \frac{-1}{s+4} - \frac{10}{(s+4)^2} + \frac{N_F(s)}{s^2+64} \equiv I_N(s) + I_F(s)$$

$$i_N(t) = -e^{-4t} - 10te^{-4t}$$

Phasor analysis

Example 13.11: (Cont.)

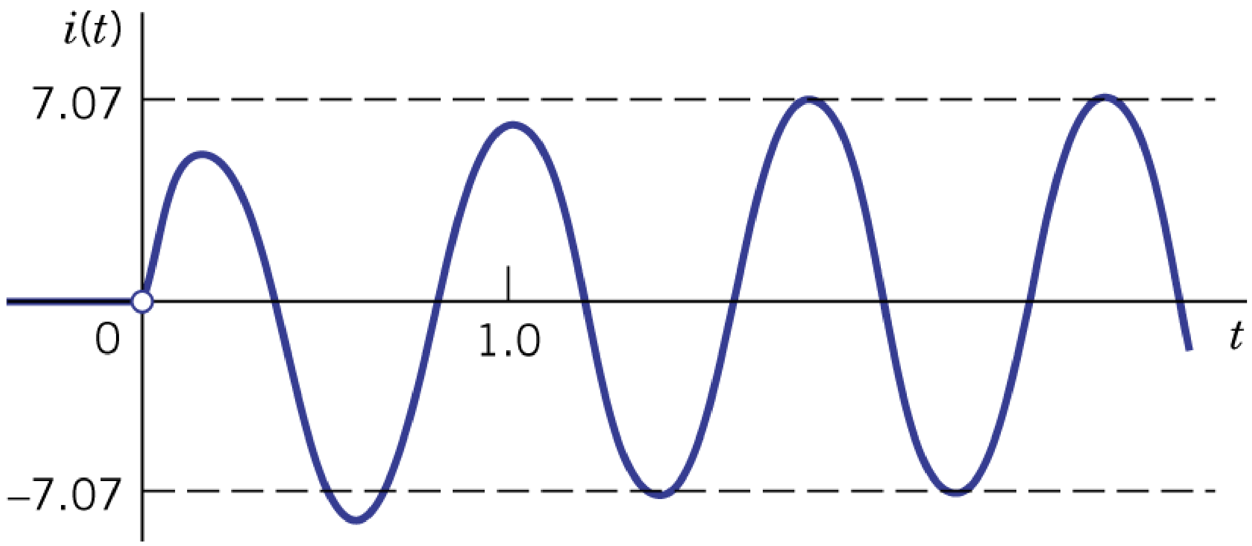
$$\underline{V}_s = 50\angle 0^0,\ H(j8) = 0.141\angle -81.9^0$$

$$\underline{I} = H(j8)\cdot \underline{V}_s = 7.07\angle -81.9^0$$

$$i_F(t) = 7.07\cos(8t - 81.9^0)$$

$$i(t) = i_N(t) + i_F(t) = -(1+10t)e^{-4t} + 7.07\cos(8t - 81.9^0),\ t \geq 0$$

$$i(0^+) = -1 + 7.07\cos(-81.9^0) = 0$$



Natural Response and Forced Response

Let $H(s) = \dfrac{1}{(s+1)(s+2)}$, $y_N(t) = A_1 e^{-t} + A_2 e^{-2t}$

$x(t) = 10e^{-s_0 t}$

case (1) : $s_0 = -3$ $(s_0 \neq -1, s_0 \neq -2)$

$$Y(s) = H(s)X(s) = \frac{10}{(s+1)(s+2)(s+3)} = \frac{5}{s+1} + \frac{-10}{s+2} + \frac{5}{s+3}$$

$y(t) = 5e^{-t} - 10e^{-2t} + 5e^{-3t}, \ t \geq 0$

(or use phasor analysis to find the forced response)

Natural Response and Forced Response (Cont.)

$$\text{case}\,(2): s_0 = -2$$

$$H(-2) \to \infty$$

$\Rightarrow$ phasor analysis is not applicable when excitation frequency is the same as a natural frequency

$\Rightarrow$ use transform analysis

$$Y(s) = \frac{10}{(s+1)(s+2)^2} = \frac{10}{s+1} + \frac{-10}{s+2} + \frac{-10}{(s+2)^2}$$

$$y(t) = \underbrace{10e^{-t}}_{\text{natural}} \underbrace{-10e^{-2t}}_{\text{mixed}} \underbrace{-10te^{-2t}}_{\text{forced}},\ t \geq 0$$

Zero-Input Response

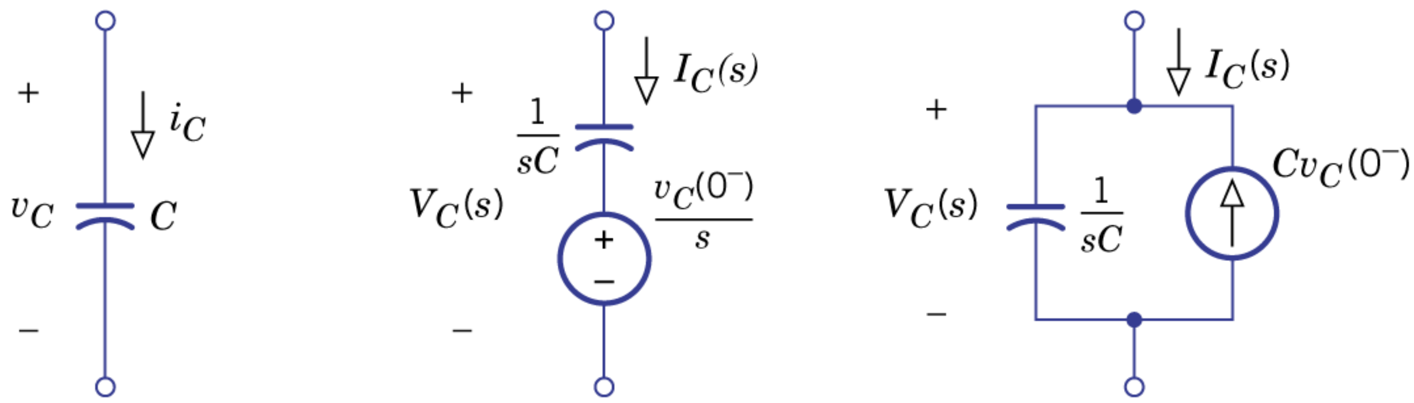
- The excitation equals zero for $t=0$ but the circuit contains stored energy at $t=0^-$.
- Thévenin/Norton equivalent circuits can be established.

Zero-Input Response (Cont.)

$$i_C(t) = Cv_C'(t)$$

$$I_C(s) = C\left[sV_C(s) - v_C(0^-)\right]$$

$$\Rightarrow \begin{cases} \text{Thevenin}: V_C(s) = \dfrac{v_C(0^-)}{s} + \dfrac{1}{sC} I_C(s) \\ \text{Norton}: I_C(s) = sCV_C(s) - Cv_C(0^-) \end{cases}$$



(a) Capacitor in the time domain

(b) s-domain Thévenin model of initial voltage

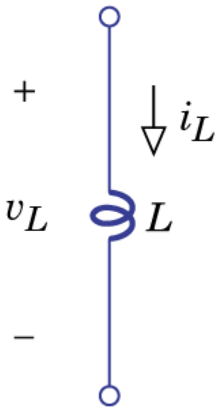
(c) s-domain Norton model of initial voltage

Zero-Input Response (Cont.)

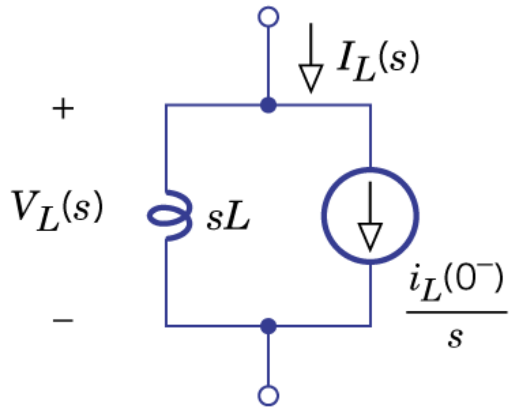
$$v_L(t) = L i_L'(t)$$

$$V_L(s) = L\left[sI_L(s) - i_L(0^-)\right]$$

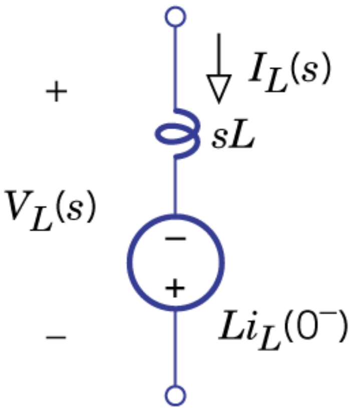
$$\Rightarrow \begin{cases} \text{Thevenin}: V_L(s) = sLI_L(s) - Li_L(0^-) \\ \text{Norton}: I_L(s) = \dfrac{i_L(0^-)}{s} + \dfrac{1}{sL} V_L(s) \end{cases}$$



(a) Inductor in the time domain

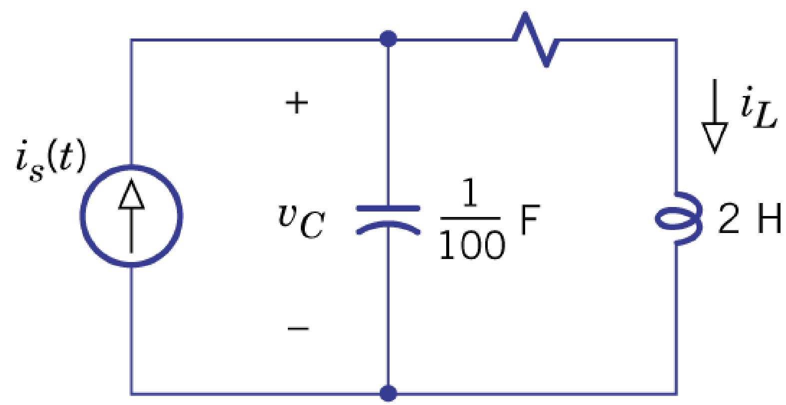


(b) s-domain Norton model of initial current

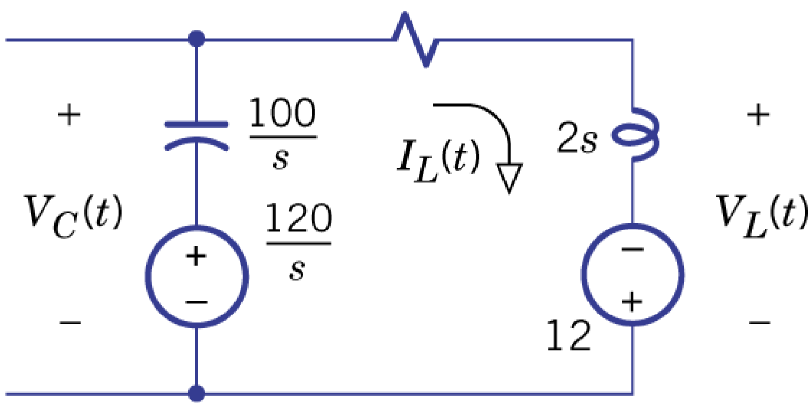


(c) s-domain Thévenin model of initial current

Example 13.12: Calculating a Zero-Input Response



(a) Circuit for Example 13.12



(b) *s*-domain diagram with initial–value sources

$$i_s(t) = 6A,\ \ t < 0$$

DC steady state analysis:

$$i_L(0^-) = 6A, v_C(0^-) = 6 \cdot 20 = 120V$$

Example 13.12: (Cont.)

zero-input response:

$$\left(2s+20+\frac{100}{s}\right)I_L(s)=12+\frac{120}{s}$$

$$I_L(s)=\frac{6s+60}{s^2+10s+50}=\frac{Bs+C}{s^2+2as+w_0^2}$$

$$\Rightarrow a=5, b=5, K=6-j6=6\sqrt{2}\angle-45^0$$

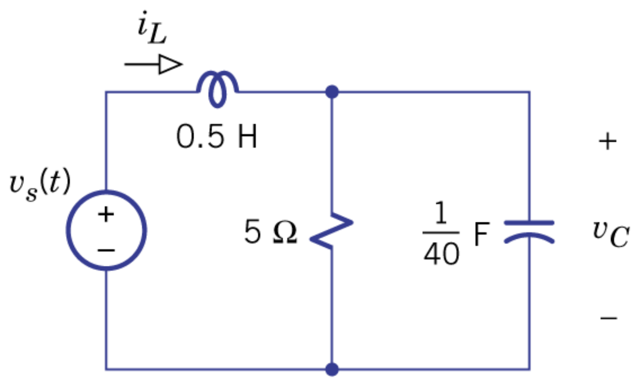
$$i_L(t)=K_m e^{-at}\cos(bt+f)$$

$$=6\sqrt{2}e^{-5t}\cos(5t-45^0)A,\ t\geq 0$$

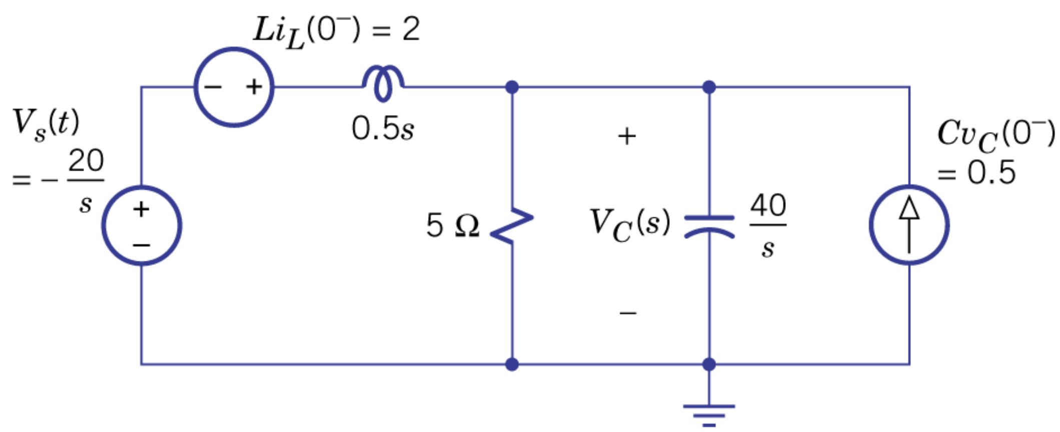
Complete Response

- Complete response: complete response=zero-input response + zero-state response (with fictitious sources)

Example 13.13: Calculating a Complete Response



(a) Circuit for Example 13.13



(b) s-domain diagram with input and initial–value sources

$$v_s(t) = \begin{cases} 20V, t < 0 \\ -20V, t \geq 0 \end{cases} \quad \text{find} \quad v_C(t)$$

DC steady state analysis for $t < 0$

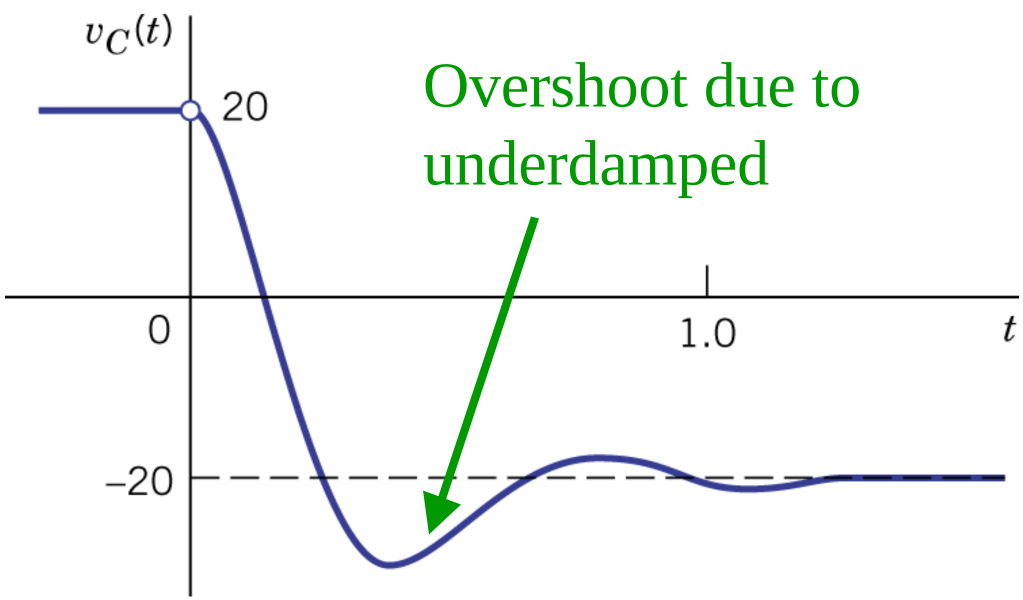
$$i_L(0^-) = 4A, \ v_C(0^-) = 20V, \ V_S(s) = -\frac{20}{s}$$

Example 13.13: (Cont.)

$$\left(\frac{s}{40}+\frac{1}{5}+\frac{1}{0.5s}\right)V_C(s)=\frac{2-20/s}{0.5s}+0.5$$

$$V_C(s)=\frac{A_1}{s}+\frac{Bs+C}{s^2+8s+80}$$

$$v_C(t)=-20+20\sqrt{5}e^{-4t}\cos(8t-26.6^0)V,\ t\geq 0$$



Chapter 13: Problem Set

- 6, 10, 16, 21, 25, 28, 31, 35, 38, 42, 46, 51, 57, 58, 63, 65